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Workshop on Hodge Theory and Algebraic Geometry

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PREFACE

This volume contains articles based on talks given at a workshop in Trento, Italy, on 4–5 September 2009, by Frédéric Campana, Paola Frediani, Bert van Geemen, Diego Matessi, and Gregory Sankaran. The diversity of the contributions reflects the wide range of investigations currently dealing with these central subjects relating classical and modern algebraic geometry, without any pretention to completeness. In addition to those mentioned above, research talks were also given by Cinzia Casagrande, Kieran O'Grady, and Christian Schnell. The work that they discussed can be found in the related references [7, 15, 16, 18].

The Workshop followed a School at which Claire Voisin and Eduard Looijenga delivered mini-courses of five lectures, and Christian Schnell gave two lectures (this time, in his capacity as course tutor). This material will appear in a future volume. The combined meeting was organized by the Centro Internazionale per la Ricerca Matematica (CIRM), as part of their successful programme for 2009.

Although not all talks addressed the first topic of the title directly, it is as well to include some introductory remarks about Hodge Theory, which was originally developed by W. V. D. Hodge in the 1930's, and is used to relate the geometry, analysis and topology of compact Kähler manifolds through the study of harmonic forms. His motivation was to solve a difficult problem posed by Severi about the vanishing of a double integral of the first kind on an algebraic surface all of whose periods are zero.

In contemporary language, Hodge Theory provides harmonic representatives for Dolbeault cohomology classes of (p,q)-forms on a compact Kähler manifold. These representatives constitute finite-dimensional vector spaces $H^{p,q}$ whose dimensions are denoted by $h^{p,q}$ and called Hodge numbers; placed together they form the Hodge diamond of the manifold. The resulting decomposition expresses each complex De Rham cohomology "group" as a direct sum of subspaces $H^{p,q}$ with the property that $H^{p,q}$ and $H^{q,p}$ are complex conjugate. This then serves as a model for an abstract Hodge structure on a vector space.

From the historical and scientific viewpoint, the theory of variation of Hodge structure, as developed by Griffiths and Deligne in the 1970's, is a powerful tool for studying algebraic varieties in characteristic zero. It consists mostly in the study of the (mixed) Hodge structures associated to algebraic varieties, and their variations, i.e. a family of Hodge structures parameterized by some space. The so-called period map allows one to study moduli spaces qualitatively, and to examine natural local systems of cohomology on them. Sometimes, it can even give rise to a uniformization of the relevant moduli spaces.

Another aspect of Hodge theory is the study of the Hodge conjecture and its variants (the generalized Hodge conjecture and the variational Hodge conjecture, to mention just two). The two aspects are related via the study of Hodge loci, which are natural subvarieties of moduli spaces. The whole subject presents a fascinating mixture of topology, Lie group theory and algebraic geometry. The best example of this is the fact that the constant local systems underlying variations of Hodge structure are of a topological nature, while the variation of Hodge structure itself and the Hodge bundles can only be defined within the framework of algebraic geometry.

Local systems and their monodromy groups actually play an important role in the articles of Paola Frediani, of Ricardo Castaño-Bernard and Diego Matessi, and of Alice Garbagnati and Bert van Geemen. In the paper of Frediani, local systems and the Gauss–Manin connection allow one to compute the second fundamental form of the period map, using appropriate exact sequences of sheaves. Castaño-Bernard and Matessi use the monodromy of the local system to predict the topology of the "lacking" fiber. In the paper by Garbagnati and van Geemen, the local system allows one to compute the Picard–Fuchs equations via the Gauss–Manin connection. As these authors write, it is important to underline the role of Hodge theory in Mirror Symmetry:

> . . . A spectacular result from Mirror Symmetry is that a certain solution of this Picard–Fuchs equation defines a power series in one variable whose coefficients a_d allow one to compute the Gromov–Witten invariants of a quintic threefold, that is, roughly, the number of rational curves of degree d on a quintic threefold.

Mirror Symmetry is based on a study of Calabi–Yau manifolds that play an important role in theoretical physics, in particular string theory, which first revealed the concept in a heuristic fashion [6]. It predicts the correct numbers of rational curves on Calabi–Yau threefolds and relates to variation of Hodge structure on the mirror family.

The concept has had profound implications within mathematics, and has led to new areas of research. An axiomatic foundation has been built through the work of Kontsevich–Manin on Gromov–Witten invariants and quantum cohomology [14], extending classical enumerative techniques in algebraic geometry. In a related direction, Castaño-Bernard and Matessi give an introduction to the so-called Gross–Siebert mirror construction. The work is implicitly related to the conjectures of homological mirror symmetry, whereby the *complex* and *symplectic* theories of mirror pairs of Calabi–Yau manifolds are interchanged [12, 13].

The article by Arie Peterson and Gregory Sankaran contains a proof of a special case of a general theorem taken from the second author's joint 2007 Inventiones paper *The Kodaira dimension of the moduli of K3 surfaces* [11], and relevant combinatorics. The moduli of K3 surfaces is closely related to spaces classifying Hodge structures, in view of the global Torelli theorem on the injectivity of the period map that associates to a K3 surface part of its Hodge structure (see Section 5). A nice account of this can be found in section 10 of [10], cited by Garbagnati–van Geemen, whose own article on the deformations of Calabi–Yau threefolds is also closely related to the K3 theory.

Campana's paper concerns a different approach to the classification of algebraic manifolds, based on properties of the canonical bundle whose triviality characterizes the K3 surfaces and Calabi–Yau manifolds mentioned above. It is therefore concerned with birational invariants and versions of the Kodaira dimension. It is related to Hodge Theory *a posteriori* in order to obtain results closer to the results of Viehweg concerning rigid families of varieties, whereby the positivity of a certain sheaf can be deduced by studying the dimension of the image of a moduli map. This theory is also related to injectivity properties of Torelli maps.

We quickly examine each of these works in turn, in the order (alphabetical by first author) in which they appear in this volume, beginning therefore with the one just mentioned.

1. *Birational stability of the cotangent bundle*

by F. Campana

This paper complements results in [3] and [5], where complete definitions can be found. Its main object of study is the so-called Kodaira dimension $\kappa(X)$ of a complex projective manifold X. This is involved in the Iitaka conjecture.

The Iitaka conjecture states that the Kodaira dimension of a fibration is at least the sum of the Kodaira dimension of the base and the Kodaira dimension of a general fiber. This motivates a classification programme for algebraic varieties, in which it is sought to represent X as a fibration over a variety of general type, with typical fiber of Kodaira dimension 0. This is quite a natural idea, given that the application of the Proj construction to the pluricanonical ring should produce a projective variety in which the sections of powers of the canonical bundle K "capture" as much as they can about X.

Also relevant is the conjecture that concerns the vanishing of the plurigenera, namely

$$\kappa(X) = -\infty \quad \Longrightarrow \quad X \text{ is uniruled.}$$

By assuming one or more selected conjectures, the author conditionally proves that *the cotangent bundle of a complex projective manifold X is "birationally" semistable, unless X is uniruled.*

The article's starting point is the concept of *saturating* a line bundle or sheaf. It contains a simple proof of the pseudo-effectivity of the relative canonical bundle of a fiber space when its generic fibers are not uniruled. Moreover, all these notions are extended to the category of "smooth orbifolds'. The paper includes a wealth of examples of Fano type.

The author writes:

> . . . Combined with Hodge theoretic arguments, it might indeed permit one to easily obtain stronger versions, closer to Viehweg's results.

One link with Hodge theory might be found in Section 8, and the discussion of the $\mathbb{Q}$-divisor D on Y, since Hodge theory deals with the representation of such rational cycles or divisors using harmonic forms.

2. *The fixed point set of anti-symplectic involutions of Lagrangian fibrations*
by R. Castaño-Bernard and D. Matessi

This paper is a review of previous results of the authors about the topology of the fixed point locus of anti-symplectic involutions on symplectic manifolds. The authors also report on some work in progress. They write:

> …one reason why the fixed point set of an anti-symplectic involution is interesting is that its Floer homology is particularly well behaved…

Many of the results and ideas mentioned here on Lagrangian fibrations are based on, or inspired by, the work of Mark Gross and Bernd Siebert.

The basic object is the so-called integral affine manifold with singularities $(B,\Delta,\mathcal{A})$. Here B is topological manifold, Δ is a closed subset of B with $B_0 := B-\Delta$ dense in B and $\mathcal{A}$ is a atlas of B_0 whose change of coordinates are (in each connected component) affine maps $Ax+v$ with $A \in Sl_n(\mathbb{Z})$. The authors explain how to attach a Lagrangian submersion $\pi_0 : X_0 \to B_0$ to such $(B,\Delta,\mathcal{A})$ and then they point out that:

> …we may ask whether we can find a symplectic manifold X and extend the bundle $f_0 : X_0 \to B_0$ to a Lagrangian fibration $f : X \to B$ by inserting singular Lagrangian fibers over the set Δ…

Notice that π_0 comes with a Lagrangian section, namely the zero section. The anti-symplectic involution τ on X_0 is the reflection $\alpha \mapsto -\alpha$.

The so-called Focus-focus example, where $B=\mathbb{R}^2$ and $\Delta=(0,0)$ is the more elementary example. This expounds the basic idea which allows one to find the "lacking" fiber over Δ. Namely, to study the fixed part of the monodromy of the local system given by the homology groups of the fibers of π_0. Also this example is very important since it is the "local model" used to extend a so-called *simple* 2-dimensional integral affine manifolds via a gluing argument. Indeed, the important example in which $B=S^2$ and Δ has 24 points is described. In this case the extended total space X is diffeomorphic to a K3 surface, and it is shown that the fixed point set Σ of the anti-symplectic involution has two connected components; one of them is the zero section and the other is a genus 10 surface.

The connection with Mirror Symmetry is explained. The link (shown by Gross) comes from the duality between the tangent and the cotangent bundle, and the quotients TB_0/Λ and $T^*B_0/\check{\Lambda}$ by the pair of dual lattices.

There is a description of Gross' results on Leray spectral sequences applied to the local systems of torus fibrations associated to simple integral affine manifolds. As a interesting consequence, it is explained why the equality $b^1(\Sigma)=20=h^{1,1}(X)$ is not a coincidence.

The article includes a discussion of the local models when $\dim B = 3$ and some comments about the deep results obtained in [2]. Finally, there is a discussion of the cohomology of one example of a fixed-point locus of an involution. This arises from the Lagrangian fibration of Schoen's Calabi–Yau manifold.

3. *The second Gaussian map for curves: a survey*
by P. Frediani

Let A_g be the moduli space of principally polarized abelian varieties of dimension g and let $j : M_g \to A_g$ be the period map sending a curve to its Jacobian. *It is an interesting and classical problem to understand the geometry of the image of M_g in A_g.*

On A_g there is a natural metric coming from the $Sp(2g,\mathbb{R})$ invariant metric (unique up to a scalar) on the Siegel space $H_g \cong Sp(2g,\mathbb{R})/U(g)$ of which A_g is the quotient by $Sp(2g,\mathbb{Z})$. An explicit expression for the second fundamental form of the immersion j was given in [9].

The second fundamental form of the immersion j is the same as that in submanifold geometry. In the present context, it is the difference between the Levi-Civita connection of the Siegel metric on A_g and the Levi-Civita connection of the induced (also called Siegel metric) on the smooth part of M_g. This is explained in [8, page 1236]. Calculations are carried out in the language of algebraic geometry using, for example, exact sequences of sheaves. This is in contrast to the approach of differential geometers, who are used to regarding the second fundamental form $\alpha(X,Y)$ of an immersion $i : M \to N$ as the normal component of the ambient covariant derivative of tangent vector fields, i.e. $\alpha(X,Y) := (\nabla^N_X Y)^\perp$.

In [9], it was proved that the second fundamental form lifts the second Gaussian map $\gamma^2_C : I_2(C) \to H^0(C, 4K_C)$ of a curve C, a result stated in an unpublished paper of Green and Griffiths. The present paper is a survey on results obtained by the author in collaboration with Elisabetta Colombo and Giuseppe Pareschi on the second Gauss map γ^2_C of a curve C, and on its relation with the second fundamental form of immersion given by the period map.

The guiding principle is that information about the map γ^2_C can be used to obtain information about the holomorphic curvature of the Siegel metric on M_g. By using the so-called Schiffer variations ξ_P, the author explains the relation between Weierstrass points of a hyperelliptic curve or a ramification point of the g^1_3 on a trigonal curve on the one hand, and the holomorphic sectional curvature $H(\xi_P)$ on the other.

The article contains a description of results on the second Gaussian map γ^2_C of a curve C that is contained in a $K3$ surface. There is also a report about both Gaussian maps γ^1_C and γ^2_C, also called Wahl maps, when C is contained in an abelian surface.

4. *Examples of Calabi–Yau threefolds parametrised by Shimura varieties*
by A. Garbagnati and B. van Geemen

The relationship of this topic with Hodge Theory is immediate via the Hodge diamond and its role in Mirror Symmetry: two mirror CY manifolds X, Y satisfy

$$h^{2,1}(X) = h^{1,1}(Y) \qquad \text{and} \qquad h^{2,1}(Y) = h^{1,1}(X).$$

At the foot of the article’s second page one finds the statement:

> …In analogy with the case of curves, abelian varieties and K3 surfaces, one studies the Hodge structures on the cohomology groups in order to understand these varieties better. For CY threefolds, only H^3 is of interest, as $H^2(X,C) = H^{1,1}(X)$.

A compact Kähler manifold X carries a Hodge structure, that is, a decomposition into types of the De Rham cohomology spaces $H^n(X,\mathbb{C})$. Moreover, such a Hodge structure is *polarized* by the intersection form Q_X. A simplest case is a genus one curve, i.e. a torus. In this case the Hodge structure determines the torus as a complex manifold or elliptic curve [19, page 169].

If X is the fiber of a deformation $\mathcal{X} \to B$, then it is natural to study the Hodge structures of the fibers in order to have an idea of the possible complex structures which X as a manifold can support. Roughly speaking, the points of the period space $\mathcal{D}$ are the candidate Hodge structures of compact complex manifolds diffeomorphic to X. Inside the period space there is subvariety, also called B, consisting of all deformations of X. In case B is the quotient of a Hermitian symmetric space of non-compact type, the deformation space is said to be a Shimura variety.

In this paper, which incorporates notes from talks of the authors, there are several explicit examples of Calabi–Yau threefolds whose deformation space is a Shimura variety.

5. *On some lattice computations related to moduli problems*

by A. Peterson and G. K. Sankaran, with an appendix by V. Gritsenko

Many moduli spaces in algebraic geometry can be described as locally symmetric varieties, i.e. quotients $\Gamma\backslash\mathcal{D}$ of a Hermitian symmetric domain $\mathcal{D}$ by an arithmetic group Γ. The problem then is to understand the birational geometry of such quotients. For example, to understand if such a quotient is of general type, meaning that $\kappa(\Gamma\backslash\mathcal{D}) = \dim(\Gamma\backslash\mathcal{D})$, where $\kappa(X)$ indicates the Kodaira dimension of X.

A compact complex surface S is a K3 surface if S is simply connected and if the canonical bundle is trivial, that is to say there exists $\omega_S \in H^2(S,\Omega^2)$ that is nowhere zero. For example, a smooth quartic in $\mathbb{CP}^3$ is a K3 surface and all quartics (modulo projective equivalence) form a (unirational) space of dimension 19.

The period of S is the point $[\omega_S]$ of the projective space $\mathbb{P}(H^2(S,\mathbb{C}))$. By the Torelli theorem, the period of a K3 surface determines its isomorphism class. The moduli space of all K3 surfaces is not Hausdorff, and for this reason it is better to restrict to moduli spaces of polarised K3 surfaces.

A polarised K3 surface of degree $2d$ is a pair (S,H) consisting of a K3 surface S and a primitive pseudo-ample divisor H on S of degree $H^2 = 2d > 0$. The moduli space of such pairs is denoted by $\mathcal{F}_{2\cdot d}$. By a result of Piatetskii–Shapiro and Shafarevich [17], $\mathcal{F}_{2\cdot d}$ is the quotient of a classical Hermitian domain of type IV, a so-called Lie ball, of dimension 19, by an arithmetic group. As such, it is a quasi-projective variety, as follows from the work of [1].

The main result of this paper is as follows:

THEOREM. The moduli space $\mathcal{F}_{2\cdot 52}$ of K3 surfaces with polarisation of degree 104 is of general type.

Its proof uses the method explained in [11], and involves a combinatorial condition:

> ...there should exist a vector l in the root lattice E_8 (or E_7 in the hyperkähler case) of square $2d$, orthogonal to very few roots...

As the authors explain, the case $d = 52$ was omitted in [11], due to an incorrect interpretation of the output of a randomised search.

An appendix of the paper by the third author explains how the case $d = 52$ could have been foreseen without the help of a computer. In any case, the paper incorporates some computer code (in the functional programming language Haskell) that was used to solve the combinatorial problem addressed in the work.

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Workshop on Hodge Theory and Algebraic Geometry

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BIRATIONAL STABILITY OF THE COTANGENT BUNDLE

Abstract. We introduce a birational invariant $\kappa_{++}(X|\Delta) \geq \kappa(X|\Delta)$ for orbifold pairs $(X|\Delta)$ by considering the Δ-saturated Kodaira dimensions of rank-one coherent subsheaves of Ω^p_X. The difference between these two invariants measures the birational unstability of $\Omega^1_{(X|\Delta)}$. Assuming conjectures of the LMMP, we obtain a simple geometric description of the invariant $\kappa_{++}(X|\Delta)$, as the Kodaira dimension of the orbifold "rational quotient" of $(X|\Delta)$.

Contents

Introduction

Roughly, we shall show, using some standard conjectures, that the cotangent bundle of a complex projective manifold X is "birationally" semi-stable, unless X is uniruled, in which case the unstability is controlled by its "rational quotient". More precisely, we introduce a birational invariant $\kappa_{++}(X) \geq \kappa(X)$, the difference measuring the birational unstability of its cotangent bundle. The invariant $\kappa_{++}(X)$ is the maximum of the "saturated" Kodaira dimensions of rank-one coherent subsheaves of all Ω^p_X, for any $p > 0$. This is a measure of the birational positivity of these subsheaves, in contrast to their "numerical" positivity, by means of polarisation slopes. Conjecturally, $\kappa_{++}(X) = \kappa(R(X))$, where $R(X)$ is the "rational quotient" of X (see Section 1). For example, one should have $\kappa_{++}(X) = -\infty$ if and only if X is rationally connected, with X unstable in this sense if and only if uniruled, but not rationally connected. The study of the Kodaira dimensions of such sheaves was initiated by F. Bogomolov in [4], where bounds and a partial geometric description of extremal cases were established.

We extend these notions and conjectures to the category of "smooth orbifolds". These appear naturally in the geometric interpretation of the saturation process of the subsheaves of $\mathrm{Sym}^N(\Omega^p_X)$ introduced in the definition of $\kappa_{++}(X)$. This category is, on the other hand, needed in an essential way for the birational classification. We then reduce the above conjecture in the "orbifold" setting to other standard conjectures of the LMMP, and to an extension of Miyaoka's generic semi-positivity for lc[1] pairs with $c_1 = 0$. The notion of rational curve, uniruledness and rational connectedness will be introduced in the context of smooth orbifolds as well. We show the uniruledness of some peculiar Fano orbifolds by specific elementary methods.

We also prove a stronger "numerical dimension" version, namely $\nu_{++}(X) = \kappa_{++}(X) = \kappa(R(X))$ in the orbifold context, conditional on the same set of conjectures, in Section 7 (see the definitions there). The proof simplifies the earlier proof given in [8] of the weaker result concerning κ_{++}.

We incidentally give in Section 8 a (seemingly) new very simple proof of the pseudo-effectivity of the relative canonical bundle of a fibre space when its generic fibres are not uniruled. This gives a weakened version of Viehweg's weak positivity results, which permits one to deduce the $C_{n,m}$ Conjecture from the Abundance Conjecture, and is potentially susceptible of further developments.

As an application outside of the birational classification, we will mention the Isotriviality Conjecture for families of canonically polarised manifolds parametrised by a "special" quasi-projective manifold, which can also be reduced to the very same set of conjectures, and thus becomes a problem in birational classification.

The present text complements results in [8], [7], and [11], where complete definitions can be found.

1. Definitions and conjectures

Let X be a complex projective connected n-fold. The main concern of birational algebraic geometry consists in deducing qualitative birational geometric properties of X from positivity or negativity properties of the canonical bundle K_X. In particular, one would like to describe in these terms the birational invariants of manifolds Y which are "rationally dominated" by X, i.e., such that there exists a dominant rational map $f : X \dashrightarrow Y$ (we then write: $Y \leq X$), and so the following invariant:.

DEFINITION 1. *Let* $\kappa_+(X) := \max_{\{Y \leq X\}} \{\kappa(Y)\}$.

Thus $n \geq \kappa_+(X) \geq \kappa(X)$, and $\kappa_+(X) = -\infty$ if and only if $\kappa(Y) = -\infty$ for any $Y \leq X$.

This invariant[2] has a conjectural description, in terms of the "rational quotient" (or "MRC-fibration") $r_X : X \dashrightarrow R(X)$. Recall that this rational fibration has rationally connected (RC, for short) fibres, and non-uniruled base $R(X)$ (or is a point if and only if X is rationally connected) [16]. When X is not uniruled, $R(X) = X$, and r_X is just the identity map.

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[1] Stands for "log canonical", see [23], for example, for this notion, as well as for klt pairs. We shall also use the standard short form LMMP for "log-minimal model program" below.

[2] We also trivially have $\kappa^+(X) \geq \kappa_+(X)$, if $\kappa^+(X)$ is the invariant defined in [10]. But conjecturally also these two invariants should always coincide. This could, in fact, be shown by the arguments used below.

CONJECTURE 1. For any X, one has $\kappa_+(X) = \kappa(R(X))$. In particular, $\kappa_+(X) = \kappa(X)$ if $\kappa(X) \geq 0$.

This conjecture can be reduced to two other quite standard conjectures. Recall that a (rational) “fibration” means here a surjective (rational) holomorphic map with connected fibres.

CONJECTURE 2. (1) (The “$C_{n,m}$ Conjecture” of Iitaka). For any fibration $f : X \to Y$, one has $\kappa(X) \geq \kappa(X_y) + \kappa(Y)$. In particular, $\kappa(X) \geq \kappa(Y)$ if $\kappa(X) \geq 0$, since then $\kappa(X_y) \geq 0$. Here X_y is the generic fibre of f.

(2) (The “Uniruledness Conjecture”). If $\kappa(X) = -\infty$, X is uniruled. (The converse is easy).

Sketch of proof of (1) *using* (2). Assume first that $\kappa(X) \geq 0$. The $C_{n,m}$ Conjecture, applied to any $f : X \to Y$, directly implies the result in this case. In general, let $r : X \to R$ be the “rational quotient”. If X is rationally connected, R is a point, and $\kappa_+(X) = -\infty$, since any $Y \leq X$ is uniruled. Thus the equality. Otherwise, let $f : X \to Y$ be any rational fibration. If the generic fibre X_r does not map to a point, Y is uniruled, and $\kappa(Y) = -\infty$. Thus $Y \leq R$ if $\kappa(Y) \geq 0$, which is the claim. □

REMARK 1. The invariant $\kappa_+(X)$ is “external” in the sense that it uses manifolds Y others than X. We shall now introduce a second, closely related, invariant, which is “internal” to X, because it refers only to data defined on X itself.

Let $f : X \to Y_p$, with $p := \dim(Y) > 0$, be a “fibration”, and let L_f be the line bundle on X defined by $L_f := f^*(K_Y) \subset \Omega^p_X$. Thus, $\kappa(X, L_f) = \kappa(Y)$, and $m.L_f \subset \mathrm{Sym}^m(\Omega^p_X)$ for all $m > 0$. We may “saturate” $m.L_f$ in $\mathrm{Sym}^m(\Omega^p_X)$ and correspondingly the space of sections of $m.L_f$, and give the following definition (for any $L \subset \Omega^p_X$, not only ones of the form L_f):

DEFINITION 2. *Let $L \subset \Omega^p_X$ a rank-one coherent subsheaf, and for any $m > 0$, let $\bar{H}^0(X, m.L) \subset H^0(X, \mathrm{Sym}^m(\Omega^p_X))$ be the subspace of sections taking values in $m.L \subset \mathrm{Sym}^m(\Omega^p_X)$ at the generic point of X. Thus $H^0(X, m.L) \subset \bar{H}^0(X, m.L)$, and $\bar{H}^0(X, m.L)$ is also the space of sections of the saturation of $m.L$ in $\mathrm{Sym}^m(\Omega^p_X)$.*

Let $\bar{h}^0 := \dim_{\mathbb{C}} \bar{H}^0$, and set

$$\kappa^*(X, L) := \limsup_{m>0} \left\{ \frac{\log(\bar{h}^0(X, m.L))}{\log(m)} \right\}.$$

By standard arguments (see [28, §5], for example), one shows that $\kappa^*(X, L)$ is either $-\infty$ or an integer at most n. A fundamental theorem of Bogomolov (see [4]) actually asserts that $\kappa^*(X) \leq p$, with equality if and only if $L = L_f$ for some dominant rational fibration $f : X \dashrightarrow Y_p$. However, Y does not need to be of general type in this situation, since due to taking saturation, $\kappa^*(X, L_f) \geq \kappa(X, f^*(K_Y)) = \kappa(Y)$, the first inequality being strict in many cases. The difference will be geometrically described below.

DEFINITION 3. *For any* X, *let* $\kappa_{++}(X) := \max_{\{p>0,\ L\subset\Omega_X^p,\ \mathrm{rk}L=1\}} \{\kappa^*(X,L)\}$.

Note that $\kappa_{++}(X)$ is a birational invariant, with $n \geq \kappa_{++}(X) \geq \kappa_+(X) \geq \kappa(X)$.

CONJECTURE 3. For any X, $\kappa_{++}(X) = \kappa_+(X) = \kappa(R(X))$.

When X is rationally connected, it is easy to see that $\kappa_{++}(X) = -\infty$ by restricting Ω_X^p to a rational curve C with ample normal bundle N, and considering its natural filtration with quotients $(\Omega_C^1) \otimes (N^*)^{\otimes k}$. A relative version of this permits one to show that $\kappa_{++}(X) = \kappa_{++}(R(X))$ for any X. See Lemma 3 below. One is thus reduced, by Conjecture 2 (2), to the special case where $\kappa(X) \geq 0$.

Notice that here, however, the case $\kappa(X) \geq 0$ cannot be derived from the $C_{n,m}$ Conjecture, since a geometric interpretation of $\kappa_{++}(X)$ is lacking. Working in a larger category will permit us at the same time to give a geometric interpretation of $\kappa^*(X, L_f)$, to formulate a suitable version of the $C_{n,m}$ Conjecture, and to give a canonical birational decomposition of any X in terms of "pure" manifolds, for which the canonical bundle has one of the three basic possible "signs" $(+,0,-)$, in some suitable birational sense.

2. Extension to the category of "smooth orbifolds"

Let $f : X \to Y_p$ be a fibration, and $L_f = f^*(K_Y) \subset \Omega_X^p$. We shall always assume that f is "neat" (i.e., that the discriminant locus of f is of snc (simple normal crossings), and that the f-exceptional divisors of X are also u-exceptional for some birational map $u : X \to X'$, with X' smooth). This condition can always be realised, by means of Raynaud (or Hironaka) flattening theorem, after suitable modifications of X and Y.

The invariant $\kappa^*(X, L_f) \geq \kappa(X, L_f) = \kappa(Y)$ can be interpreted geometrically as follows in terms of the "base orbifold" of f.

A lc pair $(X|\Delta)$ consisting of a projective manifold X and of an effective $\mathbb{Q}$-divisor $\Delta = \sum_{j\in J} a_j.D_j$, with $a_j = (1-\frac{1}{m_j})$ will be said to be "smooth" if $\mathrm{Supp}(\Delta) = \lceil\Delta\rceil$ is of snc. We shall write $a_j = (1-\frac{1}{m_j})$, or equivalently: $m_j := (1-a_j)^{-1} \in \mathbb{Q} \cap \{+\infty\}$, for the Δ-"multiplicity" of D_j (equal to 1 if D is not one of the D_j's). We shall also call such a pair a "smooth orbifold". They interpolate between the "compact" case in which $\Delta = 0$, and the "open" or "logarithmic" case, in which $\Delta = \lceil\Delta\rceil \neq 0$.

DEFINITION 4. *The "base orbifold" of* $f : X \to Y$ *is the pair* $(Y|\Delta_f)$, *with* $\Delta_f := \sum_E (1-\frac{1}{m(f,E)}).E$, *$E$ running through the set of all prime divisors of Y. We define* $m(f,E) := inf_{k\in K(E)}\{t_{k,E}\}$, *and* $t_{k,E}$ *by the equality* $f^*(E) = \sum_{k\in K(E)} t_{k,E}.D_k + R$, $K(E)$ *being the set of prime divisors* $D_k \subset X$ *such that* $f(D_k) = E$, *while R is f-exceptional.*

Notice that the sum defining Δ_f is, in fact, finite, since $m(f,E) = 1$ when E is not a component of the discriminant locus of f.

The pair $(Y|\Delta_f)$ is thought of as a “virtual ramified cover” of Y eliminating by base change the multiple fibres of f in codimension one.

The geometric interpretation of $\kappa^*(X,L_f)$ is now the following:

THEOREM 1. ([7]) *For f as above, one has*

$$\kappa^*(X,L_f) = \kappa(Y,K_Y+\Delta_f) =: \kappa(Y|\Delta_f).$$

The origin of the difference $\kappa^*(X,L_f)-\kappa(X,L_f)$ thus lies in the multiple fibres of f. This theorem completes some of the results of [4]. The study of the invariant $\kappa_{++}(X)$ thus leads to the consideration of “smooth” pairs $(X|\Delta)$, but for reasons different from the ones in the LMMP.

These “smooth pairs” can be naturally equipped with lots of geometric invariants not considered in the LMMP. We shall briefly list, but not define them[3]:

- *Morphisms and birational maps.* We thus obtain a category. If $V = X - D$ is a quasi-projective manifold, with smooth compactification X and complement D such that $(X|D)$ is smooth, then the birational class of V does not depend on the compactifying pair $(X|D)$ in this category.

- *Sheaves of symmetric differentials.* These are locally free sheaves $S^m(\Omega^p(X|\Delta))$ interpolating between $\mathrm{Sym}^m(\Omega^p_X)$ and $\mathrm{Sym}^m(\Omega^p_X(\log(\mathrm{Supp}(\Delta)))$. When $p=1$, in local analytic coordinates $(x_1,\dots,x_n)$ “adapted” to Δ (i.e., in which the support of Δ is contained in the union of coordinate hyperplanes, the hyperplane $x_j=0$ having coefficient $0\le a_j\le 1$), the sheaf $S^m(\Omega^1(X|\Delta))$ is generated, as an $\mathcal{O}_X-$module, by the elements $dx^{(N)} := \bigotimes_{j=1}^{j=n} \frac{dx_j^{\otimes N_j}}{x_j^{\lceil a_j.N_j\rceil}}$, parametrised by the n-tuples $(N)=(N_1,\dots,N_n)$ such that $m=N_1+\dots+N_n$.

In particular, $m.(K_X+\Delta)=S^m(\Omega^n(X|\Delta))$.

Morphisms $f:(X|\Delta)\to(Y|\Delta_Y)$ functorially induce maps of sheaves of symmetric differentials, moreover, the spaces $H^0(X,S^m(\Omega^p(X|\Delta)))$ are birational invariants of the smooth pair $(X|\Delta)$.

- *The “integral” case.* When Δ is moreover “integral” (i.e., if all a_j’s are of the “standard” form $a_j = 1-\frac{1}{m_j}$ with m_j either integral or $+\infty$, that is $m_j=1$), one can define additionally the 3 following invariants: $\pi_1(X|\Delta)$, the Kobayashi pseudometric $d_{(X|\Delta)}$ on X, and the notion of integral points (over any field of definition).

The invariants defined above permit one to extend, as follows, to “smooth pairs” $(X|\Delta)$ the birational invariants $n\ge\kappa_{++}(X|\Delta)\ge\kappa_+(X|\Delta)\ge\kappa(X|\Delta)$.

Let, indeed, a smooth pair $(X|\Delta)$ be given.

- For any $L\subset\Omega^p_X$, and $m>0$, let $\bar{H}^0(X|\Delta,m.L)\subset H^0(X,S^m(\Omega^p(X|\Delta)))$ be the subspace of sections taking values in $m.L\subset S^m(\Omega^p(X|\Delta))$ at the generic point of X. Equivalently, $\bar{H}^0(X|\Delta,m.L) = H^0(X,\overline{m.L}^{\Delta})$ is the space of sections of the saturation $\overline{m.L}^{\Delta}$ of $m.L$ in the sheaf $S^m(\Omega^p(X|\Delta))$.

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[3] See [7, §2] for the definitions.

- Define next

$$\kappa^*(X|\Delta,L) := \limsup_{m>0}\left\{\frac{\log(\overline{h}^0(X|\Delta,m.L))}{\log(m)}\right\}.$$

- For any neat fibration $f : X \to Y$, we define an orbifold base $(Y|\Delta_{(f|\Delta)})$ exactly as above when $\Delta = 0$, simply replacing there $m(f,E)$ by

$$m((f|\Delta),E) := \inf_{k\in K(E)}\{t_{k,E}.m_\Delta(D_k)\},$$

recalling that $m_\Delta(D_k)$ is the multiplicity of D_k in Δ. The notations are those of Definition 4 above. The reason for this definition comes from a formula to compute the orbifold base of the composition of two fibrations. When f is only rational, replace it first by a "neat" model.

Theorem 1 above still holds: $\kappa^*(X|\Delta,L_f) = \kappa(Y|\Delta_{(f|\Delta)})$.

- Define finally $\kappa_+(X|\Delta) := \max_{\{Y\leq X\}}\{\kappa(Y|\Delta_{(f|\Delta)}))\}$, and

$$\kappa_{++}(X|\Delta) := \max_{\{p>0,\ L\subset\Omega^p_X,\ \mathrm{rk}L=1\}}\{\kappa^*(X|\Delta,L)\}.$$

Conjecture 3 can now be partially extended to "smooth orbifolds":

CONJECTURE 4. For any "smooth pair" $(X|\Delta)$ such that $\kappa(X|\Delta) \geq 0$, one has $\kappa_{++}(X|\Delta) = \kappa(X|\Delta)$.

In general, we shall conjecture that $\kappa_{++}(X|\Delta) = \kappa_+(X|\Delta) = \kappa(R^*|\Delta_{(r^*|\Delta)})$, the fibration $r^* : (X|\Delta) \to R^*$, which is a substitute of the rational quotient, being conditionally defined in Proposition 2 when $\kappa(X|\Delta) = -\infty$.

We shall provide in Section 7 a conjectural geometric interpretation (see Conjecture 9 below) of the conditions $\kappa = -\infty$ and $\kappa_{++} = -\infty$ in the orbifold context, in terms of "orbifold" rational curves.

3. Orbifold additivity

Let $(X|\Delta)$ be a "smooth" pair, and $f : X \to Y$ be a "neat" fibration, with "orbifold base" $(Y|\Delta_{(f|\Delta)})$. Notice that the restriction of Δ to a generic fibre X_y of f induces a smooth pair $(X_y|\Delta_y)$.

CONJECTURE 5. ("The $C^{orb}_{n,m}$ Conjecture") $\kappa(X|\Delta) \geq \kappa(X_y|\Delta_y) + \kappa(Y|\Delta_{(f|\Delta)})$.

Observe that, even when $\Delta = 0$, this strengthens the Iitaka Conjecture $C_{n,m}$ (because of the second term on the right hand side, which takes multiple fibres into account).

THEOREM 2. ([7]) *When the orbifold base of* $(f|\Delta)$ *is of general type (i.e., if* $\kappa(Y|\Delta_{(f|\Delta)}) = \dim(Y)$), *we have:* $\kappa(X|\Delta) \geq \kappa(X_y|\Delta_y) + \dim(Y)$.

The proof is an orbifold adaptation of Viehweg's arguments used when $\Delta = 0$ ([29], see also [19] for a related result). Nevertheless, the orbifold context considerably extends the range of applicability. Applications wil be given in Section 5. We first derive some (conditional) conclusions of the conjecture.

We now introduce the two fundamental fibrations of birational classification in the orbifold context.

The first one is the Iitaka–Moishezon fibration $J : (X|\Delta) \to J(X|\Delta)$, defined by a suitable linear system $m.(K_X + \Delta)$ when $\kappa(X|\Delta) \geq 0$. Its two defining properties are that its generic orbifold fibres $(X_j|\Delta_j)$ have $\kappa = 0$, and that its base dimension is $\kappa(X|\Delta) \geq 0$.

Applying $C^{orb}_{n,m}$ to the orbifold Iitaka–Moishezon fibration gives a partial answer to Conjecture 4:

PROPOSITION 1. *Assume that the $C^{orb}_{n,m}$ Conjecture 5 holds. Then* $\kappa^*(X|\Delta, L_f) \leq \kappa(X|\Delta)$ *for any fibration* $f : X \dashrightarrow Y$.

The second fundamental fibration is a weak (conditional) version of the "rational quotient". Its existence requires assuming $C^{orb}_{n,m}$.

PROPOSITION 2. ([7]). *Assume $C^{orb}_{n,m}$. For any smooth $(X|\Delta)$, there exists a (birationally) unique fibration $r^* : (X|\Delta) \to R^* := R^*(X|\Delta)$ such that:*

(1) *Its generic orbifold fibres have* $\kappa_+ = -\infty$.

(2) *Its orbifold base has* $\kappa \geq 0$, *or is a point if and only if* $\kappa_+(X|\Delta) = -\infty$.

We now reformulate the general (conditional) version of Conjecture 4, in complete analogy with the case $\Delta = 0$:

CONJECTURE 6. Assume $C^{orb}_{n,m}$ (it is needed to define r^*). For any smooth $(X|\Delta)$, one has: $\kappa_{++}(X|\Delta) = \kappa(R^*|\Delta_{(r^*|\Delta)})$. Here $(R^*|\Delta_{(r^*|\Delta)})$ is simply the orbifold base of (any neat model of) $r^* : (X|\Delta) \dashrightarrow R^*$.

REMARK 2. Although the fibration r^* and R^* are well defined up to birational equivalence, it is not known whether its orbifold base $(R^*|\Delta_{(r^*|\Delta)})$ is uniquely defined up to birational equivalence. Its Kodaira dimension is however well defined, independently of the choices made. See [7].

We shall now reduce this fibration to a composition of fibrations of the LMMP, and Conjecture 3 to some more standard conjectures.

4. Reduction to two other conjectures

We now formulate three other conjectures, the first two ones being standard in the LMMP (due to V. Shokurov and C. Birkar).

CONJECTURE 7. Let (X,Δ) be an lc pair.

(1) There exists a sequence of divisorial contractions and flips $s : X \dashrightarrow X'$ such that if $\Delta' = s_*(\Delta)$, then either $K_{X'} + \Delta'$ is nef, or there exists a fibration $f : X' \to Y'$ with Fano, positive-dimensional orbifold fibres (X'_y, Δ'_y).

(1′) If $K_{X'} + \Delta'$ defined above is nef, it is $\mathbb{Q}$-effective. (It is a weak form of the Abundance conjecture, formulated in Conjecture 8 below).

(2) If $c_1(X|\Delta) = 0$, if $m > 0$ is an integer, and if $C = H_1 \cap \cdots \cap H_{n-1}$ is a general Mehta–Ramanathan curve on X, the restriction of $\otimes_h S^{m_h}(\Omega^{p_h}(X|\Delta))$ to C is semi-stable (i.e., all of its subsheaves have nonpositive degree), for any finite sequence (m_h, p_h) of pairs of positive integers.

The first conjecture is known for klt pairs, if the boundary Δ is assumed to be big, by [3] (see also [2, 20, 27]), the second one a special case of the Abundance Conjecture, the third one is simply the orbifold version of Miyaoka's generic semi-positivity theorem. See [26] for related arguments and considerations.

Let us give first a description of the fibration r^* using Conjectures 7 (1), (2).

DEFINITION 5. *Let $(X|\Delta)$ be a smooth projective orbifold. Define the fibration $r = r_{(X|\Delta)} : (X|\Delta) \dashrightarrow (Y|\Delta_Y)$ to be:*

(1) *The (orbifold) identity map if $\kappa(X|\Delta) \geq 0$.*

(2) *Any neat model of the composition map $r := (f \circ s) : X \dashrightarrow Y'$ of Conjecture 7 (1) if $\kappa(X|\Delta) = -\infty$, with orbifold base $(Y|\Delta_Y)$.*

Notice that neither r, nor $(Y|\Delta_Y)$ are uniquely defined, up to birational (orbifold) equivalence. Nevertheless, the composition r^n, with $n = \dim_{\mathbb{C}}(X)$, is well defined, by the following:

THEOREM 3. *Let $(X|\Delta)$ be a smooth n-dimensional projective orbifold. Assume that Conjectures 7 and $C^{orb}_{n,m}$ hold. Then $r^* = r^n$ (for any possible choice of the sequence of r's).*

Proof. Because of the uniqueness of the map r^* (up to birational equivalence), we simply need to show that, on any neat model of r^n, we have $\kappa_+ = -\infty$ for its general orbifold fibre, and $\kappa \geq 0$ for its orbifold base (on some neat model). If $\kappa(X|\Delta) \geq 0$, r^n is the identity map, and the claim is obvious (assigning $\kappa_+ = -\infty$ to orbifold points). Otherwise, let $m > 0$ be the smallest integer such that $\kappa(Y|\Delta_Y) \geq 0$, for the base orbifold of some sequence of r's of lentgh m, and of composition r^m. We then have $m \leq n$, since the dimension decreases at each of the m steps, since the intermediate orbifold bases have $\kappa = -\infty$. Now, the intermediate fibrations have generic fibres which are birational to lc Fano orbifold pairs, and have thus $\kappa_+ = -\infty$, by Lemma 2, proved below. Since a composition of rational fibrations with general orbifold fibres having $\kappa_+ = -\infty$ also has this property (by [7], 7.14), we conclude that the general orbifold fibres of r^m also have $\kappa_+ = -\infty$. ☐

A second consequence of Conjecture 7 is:

THEOREM 4. ([8, theorem 10.5]). *Assume $C^{orb}_{n,m}$ and Conjecture 7. Then Conjecture 4 also holds.*

A stronger (numerical dimension) version will be proved in detail in 7 below, along parallel lines. We thus do not give the proof again here.

5. The core and its canonical decomposition

DEFINITION 6. *We say that $(X|\Delta)$ is "special" if $\kappa^*(X|\Delta, L) < p$ for any $p > 0$ and any rank-one coherent subsheaf $L \subset \Omega^p_X$.*

Equivalently, this means that there is no fibration $f : (X|\Delta) \dashrightarrow Y$ such that its orbifold base is of general type (on any holomorphic neat model), with $\dim(Y) > 0$.

REMARK 3. Rank-one coherent subsheaves $L \subset \Omega^p_X, p > 0$ of maximum positivity (i.e., with $\kappa^*(X|\Delta, L) = p$) are called "$\Delta$-Bogomolov sheaves". Being special thus means that there are no such sheaves on $(X|\Delta)$.

Special orbifolds are natural higher-dimensional generalisations of rational and elliptic curves, with the same expected qualitative properties. They are the exact opposite of orbifolds of "general type".

COROLLARY 1 (of Theorem 2). *If $\kappa(X|\Delta) = 0$, or if $(X|\Delta)$ is Fano (i.e., if $-(K_X + \Delta)$ is ample, then $(X|\Delta)$ is special.*

By the very definition, $(X|\Delta)$ is special if $\kappa_{++}(X|\Delta) = -\infty$. It is unknown whether $\kappa_{++}(X|\Delta) = -\infty$ if $(X|\Delta)$ is Fano. This follows however from Conjecture 7 (2), as we have seen above.

The next result describes unconditionally the structure of arbitrary smooth orbifolds, in terms of its antithetic maximal parts (special "subobjects" versus general type "quotients") :

THEOREM 5 ([7, théorème 10.2]). *For any smooth pair $(X|\Delta)$, there is a (birationally) unique functorial fibration $c : (X|\Delta) \to C(X|\Delta) = C$ such that:*

(1) *Its general (orbifold) fibres are special.*

(2) *Its orbifold base is of general type (or a point, if and only if $(X|\Delta)$ is special).*

This fibration is called the "core" of $(X|\Delta)$.

REMARK 4. The fibration c is determined by the (unique) Δ-Bogomolov sheaf $L \subset \Omega^p_X$ on $(X|\Delta)$, with $p > 0$ maximum.

The second structure theorem (conditional on $C^{orb}_{n,m}$) is:

THEOREM 6 ([7, theorem 11.3]). *Assume the conclusion of Proposition 2 to be true (since it uses* $C^{orb}_{n,m}$*, we have to assume it). Then, for any smooth pair* $(X|\Delta)$*, the core map of* $(X|\Delta)$ *has the following decomposition as a composition of* $2n$ *canonically defined fibrations:* $c = (J \circ r^*)^n$.

In particular, $(X|\Delta)$ *is special if and only if it is a tower of fibrations with general orbifold fibres having either* $\kappa_+ = -\infty$*, or* $\kappa = 0$.

Notice that, even if we are only interested in the case $\Delta = 0$, non-trivial orbifold divisors will usually appear in the above decomposition.

This (conditional) decomposition often permits one to reduce the study of arbitrary manifolds to that of smooth pairs of the three basic types: $\kappa_+ = -\infty$, $\kappa = 0$, or of general type. It naturally leads to a conjectural extension of S. Lang's conjectures in arithmetics and complex hyperbolicity, for all manifolds and even smooth orbifolds. See [11] and [7] for details.

6. Numerical dimension version

Let, in this section, X be a complex projective connected, n-dimensional $\mathbb{Q}$-factorial normal space, A and D be $\mathbb{Q}$-divisors on X, with A ample.

The *numerical dimension of* D is defined as the real number

$$\nu(X,D) := \sup_{k\geq 0}\left\{\limsup_{m>0}\left\{\frac{\log(h^0(X,mD+A))}{\log(m)}\right\}\right\},$$

for $m > 0$ integral and sufficiently divisible.

Easy standard arguments show that:

1. $\nu(X,D) = -\infty$, or is real, and lies in $[0,n]$.
2. $\nu(X,D)$ does not depend on the choice of A.
3. $\nu(X,D) \geq \kappa(X,D)$.
4. $\nu(X,D) = -\infty$ if and only D is not pseudo-effective (this is one of the definitions of pseudo-effectivity).
5. When D is nef, it is an easy consequence of Kodaira vanishing and Riemann–Roch that

$$\nu(X,D) = \limsup_{m>0}\left\{\frac{\log(h^0(X,mD+A))}{\log(m)}\right\} = \nu'(X,D) \in \{0,1,\dots,n\},$$

for any ample $A = K_X + (n+2)H$, where H is any ample line bundle on X, and where $\nu'(X,D)$ is the largest integer ℓ such that $D^\ell \in H^{2.\ell}(X,\mathbb{Z})$ is not numerically zero. The Kodaira vanishing indeed says that $h^0(X,mD+A) = \chi(X,\mathcal{O}_X(mD+A))$, for any $m \geq 0$. When D is only assumed to be pseudo-effective, the Nadel vanishing theorem implies the same equality, but only after tensorising $\mathcal{O}_X(mD+A)$ with the multiplier ideal sheaf $\mathcal{I}(mD+A)$, which cannot be controlled without further ideas.

6. One may however wonder whether $\nu(X,D)$ is not an integer, if nonnegative, and if $\nu(X,D)=\nu_A(X,D)$ for A sufficiently ample (for example $A=K_X+(n+2).H$, as above). And also what is the relationship between $\nu(X,D)$ and the numerical dimension of D defined by N. Nakayama in [24] and S. Boucksom in [5].

One form of the so-called “Abundance Conjecture” is the following:

CONJECTURE 8. Assume $(X|D)$ is a “log-canonical pair”. Then

$$\nu(X,K_X+\Delta)=\kappa(X,K_X+\Delta).$$

This is known when D is “big” and $(X|D)$ is klt ([3] and [25]). This is also known when $\nu(X,K_X+\Delta)=0$ if $q(X)=0$, as follows from [24] and [5]. When $\Delta=0$, the case $q\geq 0$ follows from a more general statement in [13, §3]. When $\nu=0$, the general lc case is established (in a more general form) in [12], using the purely logarithmic case proved in [18].

PROPOSITION 3. *Assume Conjecture 8 to be true. Then Conjecture* $C^{orb}_{n,m}$ *is true.*

Proof. See [8, §10] for a proof using the weak positivity of the direct images of the orbifold pluricanonical sheaves. We give in Section 8 below a simple proof in the particular case where $\Delta=0$, using the pseudo-effectivity of $f_*(K_{X/Y})$ when X_y is not uniruled. □

We shall now state and conditionally prove a “numerical dimension” version of Theorem 4. For this we first need to define the “numerical dimension” version of κ_{++}.

DEFINITION 7. *Let* $E_. = (E_m)_{m\in\mathbb{N}^{>0}}$ *be a family of vector bundles on* X*, together with generically isomorphic bundle maps* $S^m(E_1)\to E_m$ *for any* $0<m\in\mathbb{N}$*. Let* $L\subset E_1$ *be a rank-one coherent susheaf, and let* $\overline{m.L}^{E_.}$ *be the saturation of the image of* $\mathrm{Sym}^m(L)$ *in* E_m*, for any* $m>0$.

Let A be an ample line bundle on X. We define

$$\nu_A(X|E_.,L):=\limsup_{m>0}\left\{\frac{\log(h^0(X,\overline{m.L}^{E_.}\otimes A))}{\log(m)}\right\},$$

and $\nu(X|E_.,L):=\max_{k>0}\{\nu_{kA}(X|E_.,L)\}$.

Of course, we always have:

1. $\nu(X,D)=-\infty$, or is real, and lies in $[0,n]$. Indeed $\nu(X|E_.,L)$ is bounded by the maximum dimension of the image of X by the rational map deduced from any nonzero linear system $h^0(X,\overline{m.L}^{E_.})$.

2. $\nu(X|E_.,L)$ does not depend on the choice of A.

3. $\nu(X|E_.,L)\geq\nu_A(X|E_.,L)\geq\kappa(X|E_.,L):=\limsup_{m>0}\left\{\frac{\log(h^0(X,\overline{m.L}^{E_.}))}{\log(m)}\right\}$

The main examples considered here are:

EXAMPLE 1. (1) $L = E_1$, and $E_m := m.E_1$. This is the standard case.
(2) Let $(X|\Delta)$ be a smooth orbifold, $p > 0$, and $E_m := S^m(\Omega^p(X|\Delta))$. In this case, $\nu_A(X|E_.,L)$ is denoted by $\nu_A(X|\Delta,L)$, and similarly for $\nu(X|\Delta,L)$. We also denote $\overline{m.L}^{E_.}$ by $\overline{m.L}^{\Delta}$ in this case.

DEFINITION 8. *If* $(X|\Delta)$ *is a smooth orbifold, then we define:*

$$\nu_{++}(X|\Delta) = \max_{\{p>0, L\subset\Omega^p_X\}} \nu(X|\Delta,L).$$

We now have the following strengthening of Theorem 4:

THEOREM 7. *Assume that Conjectures 8 and 7 are true. Then, for any smooth orbifold* $(X|\Delta)$*, one has* $\nu_{++}(X|\Delta) = \kappa_{++}(X|\Delta) = \kappa(R^*(X|\Delta)|\Delta_{(r^*|\Delta)})$.

Let us remark that, although the base orbifold $(R^*(X|\Delta)|\Delta_{(r^*|\Delta)})$ is not known to be birationally well defined, its canonical dimension κ is well defined.

Proof of Theorem 7. Let $(X|\Delta)$ be a smooth orbifold with X projective. Let $\mathcal{F} \subset \Omega^p_X$ be a rank-one coherent subsheaf.

Combining Conjectures 7 and 8, we first observe that Theorem 7 holds when $\kappa(X|\Delta) = 0$. Indeed we can assume, using the birational map $s : (X|\Delta) \dashrightarrow (X'|\Delta')$ provided by Conjecture 7 (1), with $c_1(X'|\Delta') = 0$, in which case the claim immediately follows from Conjecture 7 (2) by restricting to a general Mehta–Ramanathan curve $C \subset X'$, by means of Lemma 1 (1) below. From the following Lemma 1 (2), we now deduce Theorem 7 also when $\kappa(X|\Delta) \geq 0$, by using a neat model of the Moishezon–Iitaka fibration for $(X|\Delta)$.

LEMMA 1. *Let* X *be smooth projective connected, and* $E_.$ *and* $L \subset E_1$ *be as above. Let* $C = H_1 \cap \cdots \cap H_{n-1} \subset X$ *be a general Mehta–Ramanathan curve on* X *of genus* $g(C)$.

(1) *Assume that* $L.C \leq 0$*, and that* $H_i.C > A.C$*, for* $i = 1, \ldots, (n-1)$*. Then, for each ample* A *on* X*, and for any set of* $(2.g(C) + (A.C))$ *distinct points* c_k *on* C*, the natural restriction map* $H^0(X, \overline{m.L}^{E_.} \otimes A) \to \bigoplus_{k=1}^{k=2.g(C)+A.C} (m.L + A)_{|c_k}$ *is injective. In particular,* $h^0(X, \overline{m.L}^{E_.} \otimes A) \leq (2.g(C) + A.C)$ *for any* $m > 0$*, and* $\nu_A(X|E_.,L) \leq 0$.

(2) *If* $f : X \to Y$ *is a fibration such that, for any integer* $k > 0$*, there exists a bound* $B(k)$ *such that* $h^0(X_y, \overline{(m.L + k.A)}^{E_.}_{|X_y}) \leq B(k)$ *for any* $m > 0$*, then* $\nu(X|E_.,L) \leq p := \dim(Y)$.

Proof. (1) It is sufficient to show that the kernel Ker of the restriction map res : $H^0(X, \overline{m.L}^{E_.} \otimes A) \to H^0(C, \overline{m.L}^{E_.} \otimes A)_{|C}$ is zero, since the evaluation map on the points c_k is injective. But $\mathrm{Ker} = H^0(X, \overline{m.L}^{E_.} \otimes A \otimes \mathcal{I}_C)$, where $\mathcal{I}_C \cong \oplus_{i=1}^{i=(n-1)} \mathcal{O}_X(-H_i)$ is the ideal of C. Thus $\mathrm{Ker} = \{0\}$, since $(m.L + A - H_i).C < 0$ for all i's, and C belongs to an X-covering family of curves of X.

(2) It is sufficient to show that $\nu_A(X|E_.,L) \leq p$, and then to replace A by $k.A$ in the argument. Let $Z := H_1 \cap \cdots \cap H_{n-p}$ be the smooth connected complete intersection of very ample divisors on X, such that the degree of the restricted map $f_{|Z} : Z \to Y$ is at least $B(1)+1$, and $Z \cap X_y := Z_y$ consists of a $B(1)+d, d > 0$ points $c_{k,y}$ in general position on X_y. The restriction map $H^0(X_y, \overline{m.L}^{E.} \otimes A)_{X_y}) \to \bigoplus_{k=1}^{k=B(1)+d} (m.L+A)_{c_{k,y}}$ is thus injective, and so therefore is the restriction map

$$H^0(X, \overline{m.L}^{E.} \otimes A) \to H^0(Z, \overline{m.L}^{E.} \otimes A)_{|Z}).$$

Thus $\nu_A(X|E_.,L) \leq \nu_A(Z|(E_.)_{|Z}, L_Z) \leq p = \dim(Z)$. ☐

The general case will result from the following:

PROPOSITION 4. *Let $(X|\Delta)$ be smooth. Then $\nu_{++}(X|\Delta) = \nu_{++}(Y|\Delta_Y)$, if $(Y|\Delta_Y)$ is the orbifold base of any neat representative of $r^* : (X|\Delta) \to R^*(X|\Delta)$.*

By the preceding arguments, and assuming Conjecture 8, this proposition indeed implies that $\nu_{++}(X|\Delta) = \nu_{++}(Y|\Delta_Y) = \kappa(Y|\Delta_Y) = \kappa(R^*(X|\Delta)|\Delta_{(r^*|\Delta)})$, which is what Theorem 7 claims, since $\kappa(Y|\Delta_Y) \geq 0$, for $(Y|\Delta_Y)$ as in 4.

Proof (of Proposition 4). Since, by Theorem 3, we have $r^* = r^n$, for any length-n composition of rational fibrations $(f' \circ s)$ with log-Fano fibres (in the sense of the statement of Conjecture 7 (1)), it is sufficient to show that the invariant ν_{++} is preserved under such fibrations.

We first establish the statement for smooth pairs $(X|\Delta)$ which are birational to log-Fano pairs.

LEMMA 2. *Let $g : (X|\Delta) \to (X'|\Delta')$ be a birational map from the smooth orbifold $(X|\Delta)$ to the log-canonical Fano pair $(X'|\Delta')$ such that $f_*(\Delta) = \Delta'$. Assume that Conjecture 7 (2) holds. Then, for any polarisations of X', and any corresponding general Mehta–Ramanathan curve $C \subset X'$, identified with its strict transform on X, the following properties hold:*

(1) *For any finite sequence of pairs of nonnegative integers $(N_h, q_h), h = 1, \ldots, s$, and any rank-one coherent subsheaf $\mathcal{F} \subset S^{N_1}\Omega^{q_1}(X|\Delta) \otimes \cdots \otimes S^{N_s}\Omega^{q_s}(X|\Delta)$, the restriction $det(\mathcal{F})_{|C}$ has nonpositive degree at most: $-[(\sum_{h=1}^{h=s} q_h.N_h) - M.n^2]$, M being any integer such that: $-M.(K_{X'} + \Delta')$ is very ample.*

(2) $H^0(X, S^{N_1}\Omega^{q_1}(X|\Delta) \otimes \cdots \otimes S^{N_s}\Omega^{q_s}(X|\Delta) \otimes A) = \{0\}$, *for any ample line bundle A on X, if $(\sum_{h=1}^{h=s} q_h.N_h) > M.n^2 + A.C$.*

(3) $h^0(X, \overline{m.L}^{\Delta} \otimes A) = 0$ *if $m.(\sum_{h=1}^{h=s} q_h.N_h) > M.n^2 + A.C$.*

(4) $\nu_{++}(X|\Delta) = -\infty$.

Proof. (1) Let $\mathcal{G} := S^{N_1}\Omega^{q_1}(X|\Delta) \otimes \cdots \otimes S^{N_s}\Omega^{q_s}(X|\Delta)$. Let $H' = \sum_{j=1}^{j=n} H_j$, where the H_j's are general members of $M.(-(K_{X'} + \Delta'))$, $M > 0$ being a sufficiently large integer, the H_j's being chosen so that $(X'|\Delta' + \frac{1}{mn}.H') := (X'|\Delta")$ is log canonical, with

$(K_{X'}+\Delta")$ trivial on X', and such that $\Delta^+ := (\Delta+\frac{1}{mn}.H)$ has normal crossings support, H being the strict transform of H' in X. Choose $C\subset X'$ meeting H' transversally, but not meeting the indeterminacy locus of g^{-1}, and so identified with its strict transform on X. Then $L\subset \mathcal{G}^+ := S^{N_1}\Omega^{q_1}(X|\Delta^+)\otimes\cdots\otimes S^{N_s}\Omega^{q_s}(X|\Delta^+)_{|C}$ has nonpositive degree, by Conjecture 7 (2), since it is a rank one subsheaf of the locally free sheaf $\mathcal{G}^+$, assumed to be semi-stable, and with trivial determinant.

Assume now that the H_j's have been chosen in such a way that they build a system of coordinate hyperplanes for suitable local coordinates at a generic point $a\in X'$ outside of the support of Δ' and the indeterminacy locus of g^{-1}, and belonging to the smooth locus of X'. We choose also C in such a way that $a\in C$. The natural inclusion $\mathcal{G}\subset\mathcal{G}^+$ now vanishes at order at least $[(\sum_{h=1}^{h=s} q_h.N_h)-M.n^2]$ at a, as follows from lemma 3.3 of [8], since $q_h\leq n$, for any $h=1,\dots,s$. It follows that the degree of L on C is nonpositive, and is at most $-[(\sum_{h=1}^{h=s} q_h.N_h)-M.n^2]$, and so that it is negative if $(\sum_{h=1}^{h=s} q_h h.N_h)>M.n^2$.

(2) This follows from the fact that the restriction of such a section to C vanishes, unless $N_h.q_h=0, h=1,\dots,s$, since a nonzero section would otherwise generate a (locally free) rank-one coherent sheaf of negative degree on C, by the previous estimate on the degree of $\mathcal{F}_C$. The last two assertions are now obvious. ☐

We shall now deal with the (rational) fibrations having log-canonical Fano fibres. Let us first describe the situation provided by Conjecture 7 (1), assuming that $\kappa(X|\Delta)=-\infty$. Applying Conjecture 7 (1), we get a birational map $s:(X|\Delta)\dashrightarrow(X'|\Delta')$ and a log-Fano fibration $f:(X'|\Delta')\to Y$, with $\dim(Y)<n$ and $(X'_y|\Delta'_y)$ log-canonical and Fano for generic $y\in Y$. We can moreover assume that $s:(X|\Delta)\to(X'|\Delta')$ is regular and a log resolution, and also that $f\circ s:(X|\Delta)\to(Y|\Delta_Y)$ is a neat orbifold morphism, by making a suitable orbifold modification of $(X|\Delta)$ and choosing appropriate multiplicities on the divisors of X which are $f\circ s$-exceptional.

We are thus in the position to apply the following Lemma 3, which implies the claim, and thus Proposition 4 and Theorem 7. ☐

LEMMA 3. *Let $(X|\Delta)$ be a smooth orbifold, and $f:X\to Y$ be a neat fibration which is an orbifold morphism with generic smooth orbifold fibre $(X_y|\Delta_y)$ and smooth orbifold base $(Y|\Delta_Y:=\Delta_{(f|\Delta)})$.*

(1) *Assume that, for any finite sequence of pairs of positive integers (N_h,q_h) with $h=1,\dots,t$, one has: $H^0(X_y,S^{N_1}\Omega^{q_1}(X_y|\Delta_y)\otimes\cdots\otimes S^{N_t}\Omega^{q_t}(X_y|\Delta_y))=\{0\}$. Then $f_*(S^N\Omega^q(X|\Delta))=S^N\Omega^q(Y|\Delta_Y)$, for any integer $N>0$ and $q>0$.*

(2) *Assume, additionally, that $\nu_{++}(X_y|\Delta_y)=-\infty$. Then, we also have: $\nu_{++}(X|\Delta)=\nu_{++}(Y|\Delta_Y)$.*

Proof. (1) This is just lemma 4.23 of [8]. (The statement given there is global on X, but its proof applies locally over Y).

(2) For any ample line bundle A on X, and any pair (N,q) of positive integers, we thus have $H^0(X,S^N\Omega^q(X|\Delta)\otimes A)\cong H^0(Y,S^N\Omega^q(Y|\Delta_Y)\otimes f_*(A))$. Assume that some rank-one coherent subsheaf $\mathcal{F}\subset\Omega^q_X$ is such that $\nu_A(X|\Delta,\mathcal{F})\geq 0$.

We shall prove first that there exist $\mathcal{G}\subset\Omega^q_Y$ such that, generically over Y, $\mathcal{F}=f^*(\mathcal{G})$. Otherwise, there would exist a largest $s>0$, such that $\mathcal{F}$ had nonzero image $\mathcal{F}_y$ in the quotient $f^*(\Omega_Y^{(q-s)})\wedge\Omega^s_{X_y}\cong(\Omega^s_{X_y})^{\oplus R}, R:=\binom{q-s}{p}$, of the graduation of the natural filtration of $\Omega^q_{X|X_y}$ determined by f on its generic orbifold fibre $(X_y|\Delta_y)$ (see [8, §4]). By the assumption that $\nu_A(X|\Delta,\mathcal{F})\geq 0$, there are arbitrarily large integers m such that $\overline{(m.\mathcal{F})}^{\Delta}\otimes A$ has a nonzero section. But these sections would induce by projection nonzero sections of $S^m\Omega^s(X_y|\Delta_y)\otimes A_{X_y}$ contained in $\overline{(m.\mathcal{F}_y)}^{\Delta_y}$, contradicting the hypothesis that $\nu_{++}(X_y|\Delta_y)=-\infty$.

From the preceding arguments, we deduce that for any $m>0$ we have: $h^0(X,\overline{(m.\mathcal{F})}^{\Delta}\otimes A)=h^0(Y,\overline{(m.\mathcal{G})}^{\Delta_Y}\otimes f_*(A))$. Let now B be an ample line bundle on Y. Since there exists positive integers k and r such that the sheaf $f_*(A)$ can be embedded in $(k.B)^{\oplus r}$, we see that $h^0(Y,\overline{(m.\mathcal{G})}^{\Delta_Y}\otimes f_*(A))\leq r.h^0(Y,\overline{(m.\mathcal{G})}^{\Delta_Y}\otimes(k.B))$. Thus $\nu_A(X|\mathcal{F})\leq\nu_{kB}(Y|\Delta_Y,\mathcal{G})$. Since this holds for any A, the lemma is established. ☐

7. Orbifold rational curves

We shall now provide a conjectural geometric interpretation (see Conjecture 9 below) of the conditions $\kappa=-\infty$ and $\kappa_{++}=-\infty$ in the orbifold context, in terms of "orbifold" rational curves. This interpretation is entirely similar to the case when $\Delta=0$, once the notion of orbifold rational curves is defined.

DEFINITION 9 ([7, §6]). *Let $(X|\Delta)$ be a smooth orbifold, with $\Delta:=\sum_{j\in J}(1-\frac{1}{m_j}).D_j$. Let C be a smooth connected projective curve. A map $g:C\to(X|\Delta)$ is a Δ-rational (resp. a Δ-elliptic) curve if:*

(1) *It is birational onto its image, which is not contained in* $\mathrm{Supp}(\Delta)$.

(2) $\deg(K_C+\Delta_g)<0$ *(resp.* $\deg(K_C+\Delta_g)=0$*), where Δ_g is the orbifold divisor on $\mathbb{P}^1$ which assigns to any $a\in\mathbb{P}^1$ the multiplicity* 1 *if $g(a)\notin$* $\mathrm{Supp}(\Delta)$, *and otherwise the multiplicity $m_g(a):=\max_{j\in J(a)}\{\max\{1,\frac{m_j}{t_{j,a}}\}\}$.*

Here $J(a):=\{j\in J|g(a)\in D_j\}$, and $t_{j,a}$ is the order of contact of g and D_j at a, defined by the equality: $g^*(D_j)=t_{j,a}.\{a\}+\cdots$, if $j\in J(a)$.

Notice that $C\cong\mathbb{P}^1$ if g is Δ-rational, but that C is either rational or elliptic if g is Δ-elliptic. If C is elliptic, it is Δ-elliptic if and only if $g(C)$ does not meet the support of Δ.

There is also a stronger "divisible" version of these notions, which we shall not define here.

EXAMPLE 2. (1) $\Delta_g=0$, and so g is a Δ-rational curve if $t_{j,a}\geq m_j$ for any $a\in\mathbb{P}^1$ and $j\in J(a)$. A special case is when $C\subset X$ is a rational curve meeting each of the D_j in distinct smooth points of $\mathrm{Supp}(\Delta)$, each with multiplicity m_j. In this case, $\Delta_g=0$. Such a rational curve will be said "Δ-nice" in the sequel. In this case, m_j must divide $D_j.C$, for each $j\in J$.

(2) If $\Delta = \mathrm{Supp}(\Delta)$, so logarithmic, a rational curve on X is Δ-rational (resp. Δ-elliptic) if and only if its normalisation meets Δ in at most one point (resp. in exactly two points).
(3) If $\Delta' \leq \Delta$, then any Δ-rational (resp. Δ-elliptic) curve is also Δ'-rational (resp. either Δ'-rational or Δ'-elliptic) .

EXAMPLE 3. (Orbifold-ramified covers). Let $u : X' \to (X|\Delta)$ be a surjective finite ramified cover, with X' smooth connected and $(X|\Delta)$ smooth. We say that u is "orbifold ramified" if the m_j's are integers, if u is unramified over $X - \mathrm{Supp}(\Delta)$, and if, for any $j \in J$, $u^*(D_j) = \sum_k m'_{j,k}.D'_{j,k}$ are such that m_j divides $m_{j,k}$ for any j,k. This cover is "orbifold étale" if $m_j = m_{j,k}$, for every j,k.

In general, orbifold-ramified covers do not exist. An example is nevertheless the following: $u : \mathbb{P}^n \to (\mathbb{P}^n|\Delta)$, where $\Delta = \sum_{j=0}^{j=n}(1 - \frac{1}{m_j}).H_j$, the H_j being the $n+1$ coordinate hyperplanes.

We have then the following result ([7, théorème 6.33]): assume that $u : X' \to (X|\Delta)$ is an orbifold-ramified cover. Let $C' \subset X'$ be a rational curve not contained in $u^{-1}(\mathrm{Supp}(\Delta))$. Then $C := u(C') \subset X$ is a ("divisible") Δ-rational curve. Conversely, if $C \subset X$ is a ("divisible") Δ-rational curve, any component of $C' := u^{-1}(C)$ is rational if u is orbifold étale.

DEFINITION 10. *The smooth pair $(X|\Delta)$ is uniruled (resp. rationally connected) if and only if any generic point of X (resp. any generic pair of points of X) is contained in some Δ-rational curve.*

EXAMPLE 4. Let $(\mathbb{P}^n|\Delta)$, where $\Delta = \sum_{j=0}^{j=n}(1 - \frac{1}{m_j}).H_j$, the H_j being the $n+1$ coordinate hyperplanes be as in Example 3. Then $(\mathbb{P}^n|\Delta)$ is rationally connected, since $\mathbb{P}^n$ is rationally connected (in the usual sense).

We finish by a last conjecture, which extends to the orbifold context the Uniruledness Conjecture, which corresponds to the case $\Delta = 0$:

CONJECTURE 9. Let $(X|\Delta)$ be a smooth pair, then:
(1) $\kappa(X|\Delta) = -\infty$ if and only if $(X|\Delta)$ is uniruled.
(2) $\kappa_{++}(X|\Delta) = -\infty$ if and only if $(X|\Delta)$ is rationally connected.

Of course, one would like to extend to the orbifold setting the many facts known when $\Delta = 0$. But very few is known in this direction. For example, it is even unknown whether Fano smooth pairs are uniruled, this even in dimension 2 (but the logarithmic case is then true, by [20]). The case of Fano orbifolds is the decisive case for the solution of Conjecture 9, by the Structure Theorem 3, which birationally expresses orbifolds with $\kappa_+ = -\infty$ as towers of fibrations with Fano orbifold fibres.

REMARK 5. We shall give a counting argument supporting the uniruledness of Fano smooth orbifolds, by showing that covering families of Δ-nice rational curves (see Example 2 for this notion) should exist in this situation. Let indeed $g_0 : \mathbb{P}^1 \to X$ be a nonconstant map with $C := g_0(\mathbb{P}^1) \subsetneq \mathrm{Supp}(\Delta)$, going through a general point $a \in X$. Assume that $D_j.C = k_j.m_j$, for each $j \in J$, with k_j an integer. The variety $\mathrm{Hom}_a(\mathbb{P}^1, X)$ of such maps has at g_0 dimension $\dim_{g_0} \mathrm{Hom}_a(\mathbb{P}^1, X) \geq -K_X.C + 3$. The number of conditions for C to have order of contact at least m_j at an (undetermined) point of D_j lying on C is equal to $m_j - 1$. The total number of conditions for g_0 to be "Δ-nice" is thus $\sum_j k_j.(m_j - 1) = \sum_j (1 - \frac{1}{m_j}).k_j.m_j = \sum_j (1 - \frac{1}{m_j}).D_j.C = \Delta.C$.

The expected dimension of the variety of such "Δ-nice" rational curves through a is thus at least $-(K_X + \Delta).C + 3$, which thus remains positive after forgetting the 3-dimensional space of parametrisations of $\mathbb{P}^1$, precisely when $(X|\Delta)$ is Fano.

EXAMPLE 5. Let us consider the case when $X = \mathbb{P}^n, n \geq 2$, and when the support of Δ consists of k hyperplanes H_j in general position, with finite *integral* multiplicities $(m_0, \dots, m_{k-1})$, with $2 \leq m_0 \leq m_1 \leq \dots \leq m_{k-1}$, in which case we shall just say that $(\mathbb{P}^n|\Delta)$ is of type $(m_0, \dots, m_{k-1})$. The condition that $(\mathbb{P}^n|\Delta)$ be Fano (resp. has trivial canonical bundle) is then just that $\sum_j (1 - \frac{1}{m_j}) < (n+1)$ (resp. that: $\sum_j (1 - \frac{1}{m_j}) = (n+1)$). The Fano condition is thus always satisfied when $k \leq (n+1)$. When $k \leq (n+1)$ it is not difficult to see (see [7], and Example 3), that, for any finite set of points of $\mathbb{P}^n$, none of them lying on Δ, there is an irreducible Δ-rational curve[4] containing these points. See Example 6 below for a direct proof of their uniruledness. By contrast, when $k = (n+2)$, it is not known whether these Fano orbifolds are rationally connected. We shall give examples in which it can be shown by specific methods that they are, at least Δ-uniruled (and covered by Δ-elliptic curves when their canonical bundle is trivial). Observe that, for any $n \geq 2$, there is anyway only a finite number of $(n+2)$-tuples of integers $(m_0, \dots, m_{n+1})$ such that $\sum_{j=0}^{n+1} (1 - \frac{1}{m_j}) \leq (n+1)$ (of course, provided that $\sum_{j=0}^{n} (1 - \frac{1}{m_j}) \leq n$, see Proposition 6 below for this finiteness statement and examples).

EXAMPLE 6. Assume first that $(\mathbb{P}^n|\Delta)$ is of type $(m_0, \dots, m_k)$, with $k \leq n$. Then the H_j's, for $j = 1, \dots, k-1$ intersect in a projective space P of dimension $n - (k-1) \geq 0$. Any projective line meeting P, but not contained in P, meets the support of Δ in at most 2 points, and has finite Δ_g multiplicities (m_0 and m_{k-1}) there, and is thus Δ-rational, and $(\mathbb{P}^n|\Delta)$ is thus uniruled by the family of lines through P.

EXAMPLE 7. Assume next that $(\mathbb{P}^n|\Delta)$ is of type $(m_0, \dots, m_{n+1})$, and that any smooth orbifold $(\mathbb{P}^{n-1}|\Delta')$ of type $(m_0, \dots, m_{n-2}, m_{n-1}, m_{n+1})$ is Δ'-uniruled (just m_n has been omitted) if $\sum_{j \neq n} \frac{1}{m_j} > 1$. Then $(\mathbb{P}^n|\Delta)$ is Δ-uniruled if $\sum_{j \neq n} \frac{1}{m_j} > 1$. Indeed, the generic member of the pencil of hyperplanes H_s containing $P := H_n \cap H_{n+1}$ is naturally equipped with the orbifold structure $\Delta_s := \sum_{j \neq n} (1 - \frac{1}{m_j}).H_{s,j}$, where $H_{s,j} := H_J \cap H_s$, so that $H_{s,n+1} = P$. Moreover, for each curve $g : C \to H_s$ birational onto its image, not contained in the union of the $H_{s,j}$, the orbifold divisor on C computed from $(H_s|\Delta_s)$ and $(\mathbb{P}^n|\Delta)$ coincide (this is an immediate check). Assuming that $\sum_{j \neq n} \frac{1}{m_j} > 1$, we deduce from the assumption made, that H_s is Δ_s-uniruled. Thus $(\mathbb{P}^n|\Delta)$ is uniruled,

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[4] And even a "divisible" one, see [7] for this notion.

too. (The same statement should hold, assuming that $\sum_{j\neq n+1}\frac{1}{m_j}>1$, but no obvious geometric construction seems to give this.)

EXAMPLE 8. The first case when $k=n+2$ is when $X=\mathbb{P}^2$, and $\Delta=\sum_{j=0}^{j=3}(1-\frac{1}{m_j}).D_j$ is supported on 4 lines in general position, of multiplicities (m_0,m_1,m_2,m_3). Then $(X|\Delta)$ is Fano if and only if $\sum_j\frac{1}{m_j}>1$. This is the case when $(m_0,m_1,m_2,m_3)=(2,3,7,41)$, for example. It is then easy to show that a line is a Δ-rational curve if and if only if it goes through 2 of the 6 double points of the union of the 4 lines, so there are only 15 such lines. But there is a one-dimensional family of conics which are Δ-rational: the conics C which are tangent to each of the 4 lines. Indeed, for such a generic smooth conic $g:C\to\mathbb{P}^2$, g being the incusion, the divisor Δ_g is supported on the 4 points of tangency with multiplicities $(\frac{m_0}{2},\frac{m_1}{2},\frac{m_2}{2},\frac{m_3}{2})$, which is an orbifold rational curve, since

$$-2+(1-\frac{2}{m_0})+(1-\frac{2}{m_1})+(1-\frac{2}{m_2})+(1-\frac{2}{m_3})=2.[1-(\frac{1}{m_0}+\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3})]<0.$$

This Fano orbifold is thus indeed at least uniruled. By the very same argument, it is covered by Δ-elliptic conics when its canonical bundle is trivial.

We shall now partially extend the preceding example to higher dimensions.

EXAMPLE 9. Let us consider the case when $X=\mathbb{P}^n, n\geq 2$, and when the support of Δ consists of $(n+2)$ hyperplanes $H_j, j=0,1,\dots,(n+1)$ in general position (i.e., such that the intersection of any $(n+1)$ of them is empty), with finite multiplicities $(m_0,\dots,m_{n+1})$, such that $\sum_j(1-\frac{1}{m_j})<(n+1)$, or equivalently, that $\sum_j(\frac{1}{m_j})>1$. We can, and shall, assume that the first $(n+1)$ hyperplanes $H_j, j=0,\dots,n$ are the coordinate hyperplanes of equations $X_j=0$, and that the last hyperplane H_{n+1} has equation $X_0+\dots+X_n=0$, since all $(n+2)$-tuples of hyperplanes in general position are equivalent under homographies.

Remark. Assume that $C\cong\mathbb{P}^1$ is the normalisation of an irreducible rational curve of degree d in $\mathbb{P}^n$, meeting each of the hyperplanes H_j in a single point a_j, thus with contact order d. Let us try to determine a condition on d which implies that C is Δ-rational. The orbifold multiplicity at such a point $a_j\in C$ is thus $\frac{m_j}{d}$. The corresponding orbifold divisor on C thus consists of $(n+2)$ points with multiplicities $\frac{m_j}{d}$, and this curve is thus Δ-rational if and only if $\sum_j\frac{d}{m_j}>(n+2)-2=n$, equivalently, if $\sum_j\frac{1}{m_j}>\frac{n}{d}$. Since, by assumption, $\sum_j\frac{1}{m_j}>1$, this condition is realised as soon as $\frac{n}{d}\leq 1$. We thus may choose $d=n$, and C to be a rational normal curve of degree n. Notice, however, that we did not take into account the fact that the multiplicity at a_j is taken to be 1, and not $\frac{m_j}{d}<1$ if $m_j<d$. This is discussed in Example 10 below.

Recall that a normal rational curve of degree n on $\mathbb{P}^n$ is parametrically given by $P(t)=(P_0(t):\ldots:P_n(t))$, where the $P_j(t)$'s are linearly independent polynomials of degree n. All such curves are equivalent under the natural action of $\mathbb{P}Gl(n+1,\mathbb{C})$.

THEOREM 8. *For any set of* $(n+2)$ *hyperplanes* H_j *in general position on* $\mathbb{P}^n$*, and for any* $p=(p_0:\ldots:p_n)\in\mathbb{P}^n$ *generic, there exists a rational normal curve* C *of degree* n *on* $\mathbb{P}^n$*, which goes through* p*, and meets each of the* $(n+2)$ *hyperplanes* H_j *in exactly one point, which lies on the smooth part of the union of the* H_j*'s.*

Observe however that such a curve is "virtually Δ-rational", but not always Δ-rational if $n\geq 3$ (see Example 10 below).

Proof. The condition that C meets each of the coordinate hyperplanes H_j in a single point thus reads as C being given by a parametric representation of the form: $P(t)=(b_0.(t+a_0)^n,\ldots,b_n.(t+a_n)^n)$ for nonzero and pairwise distinct complex numbers a_j, and nonzero complex numbers b_j, $j=0,\ldots,n$.

The condition that C meets the $(n+2)$-th hyperplane H_{n+1} translates into the equation: $b_0.(t+a_0)^n+\cdots+b_n.(t+a_n)^n=b.(t+a)^n$ for nonzero numbers $b=\sum b_j$ and a, the latter being distinct from all of the preceding a_j's.

Because we also want C to go through the point $p=(p_0,\ldots,p_n)$, at time $t=\infty$ say, we get the additional conditions $b_j=p_j$.

In our situation, the p_j's are given, and the numbers a,b_j,a_j are to be determined from these algebraic equations. These equations are however of high degree and difficult to solve. We shall proceed differently: solve for the b_j's assuming a and the a_j's to be given, because this is simply a linear system. And then show that a and the a_j's exist for a generic choice of the p_j's, after projectivisation. This is sufficient to imply the result, by choosing the p_j's generically.

Writing the $(n+1)$ coefficients of the two polynomials in t on the two side of the equation $b_0.(t+a_0)^n+\cdots+b_n.(t+a_n)^n=b.(t+a)^n$ above permits us to rewrite this equation as a linear system in the p_j's. In matrix form, it reads as $A.v=b.w$, with A the following square complex matrix of size $(n+1)$:

$$A:=\begin{pmatrix} 1 & 1 & \cdots & 1 & 1\\ a_0 & a_1 & a_2 & \cdots & a_n\\ a_0^2 & a_1^2 & a_2^2 & \cdots & a_n^2\\ \cdots & \cdots & \cdots & \cdots & .\\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix},$$

${}^tv:=(p_0,\ldots,p_n)\in(\mathbb{C}^*)^{n+1}$, and ${}^tw:=(1,a,a^2,\ldots,a^n)\in(\mathbb{C}^*)^n$.

Now Cramer's rule allows us to express the $p_j=v_j$ as a quotient of two determinants of Vandermonde type, in terms of the b,a,a_j supposed to be given. We finally get, after some simplification: $v_j:=p_j=b.\prod_{h\neq j}\left(\frac{a-a_h}{a_j-a_h}\right)$, h running from 0 to n, avoiding j. Observe that the right hand side is invariant by translation (i.e., takes the same value when we replace a,a_j by $a+t,a_j+t$, for any $t\in\mathbb{C}$).

We are thus reduced, by the above translation invariance, which permits us to take $a=0$, to prove that the following rational map Φ is dominant.

The map $\Phi : \mathbb{C}^{n+1} \dashrightarrow \mathbb{P}^n$ is defined by $\Phi(a_0,\dots,a_n) := [x_0 : \dots : x_n]$, with

$$x_j := \prod_{h\neq j} \left(\tfrac{a_h}{a_h-a_j}\right) = \prod_{h\neq j} \left((1-(\tfrac{y_h}{y_j}))^{-1}\right) = \Psi(y_1,\dots,y_n),$$

replacing the $(n+1)$ variables $a_0,\dots,a_n$ by the n variables $y_j := \frac{a_0}{a_j}$, for $j = 1,\dots,n$. We denote also for notational simplication $y_0 := \frac{a_0}{a_0} = 1$.

Thus Φ is dominant if the determinant $\det(J)$ of the Jacobian matrix of the logarithmic derivatives of the n functions $u_j := \frac{x_j}{x_0}$ does not vanish at some point where the functions $u_j = u_j(y_1,\dots,y_n)$ are regular and nonzero.

Let us first rewrite, after some simplification, $u_j := -y_j^n . \prod_{h>0,h\neq j} \left(\frac{1-y_h}{y_j-y_h}\right)$.

Taking logarithmic derivatives now shows that:

- $\frac{y_k}{u_j}.\frac{\partial u_j}{\partial y_k} = -\frac{(1-y_j)}{(1-y_k)(1-\frac{y_j}{y_k})}$) if $1 \leq j \neq k \geq 1$, and that
- $\frac{y_j}{u_j}.\frac{\partial u_j}{\partial y_j} = n - \sum_{h\neq j} \left(1-\frac{y_h}{y_j}\right)^{-1}$.

Let us now choose $M > 0$, and the y_j's all nonzero in such a way that $|y_j| > M.|y_{j-1}|$ for $j = 2,\dots,n$, with $M.|y_n| < 1$. As M tends to $+\infty$, one easily checks, using the equalities above, and since y_j/y_k tends to 0 if $j < k$ and to ∞ if $j > k$, that

- $\frac{y_k}{u_j}.\frac{\partial u_j}{\partial y_k}$ tends to -1 if $j < k$, and tends to 0 if $j > k$, while:
- $\frac{y_j}{u_j}.\frac{\partial u_j}{\partial y_j}$ tends to $n-(j-1)$, for any $j = 1,\dots,n$.

Thus $\det(J).y_1.\cdots.y_n$ tends to $\det(J_0)$, where J_0 is the matrix with coefficients -1 below the diagonal, with coefficients 0 above the diagonal, and with coefficient $n-(j-1)$ on the diagonal, at the intersection of the j-th line and j-th row, for $j = 1,\dots,n$. Since $\det(J_0) = n! \neq 0$, $\det(J) \neq 0$ when the y_j's satisfy the above inequalities for M sufficiently large , which implies the desired assertion that Φ is dominant. ☐

From Theorem 8 above we shall now deduce that smooth orbifolds are covered by "virtual" Δ-rational or elliptic curves, according to whether they are Fano, or have trivial canonical bundle. We define first these "virtual" notions.

DEFINITION 11. *Let $(X|\Delta)$ be a smooth orbifold, with $\Delta := \sum_{j\in J}(1-\frac{1}{m_j}).D_j$. Let C be a smooth connected projective curve. A map $g : C \to (X|\Delta)$ is a "virtual" Δ-rational (resp. a "virtual" Δ-elliptic) curve if:*

(1) *It is birational onto its image, which is not contained in* $\mathrm{Supp}(\Delta)$.

(2) $\deg(K_C+\Delta_g^*) < 0$ *(resp.* $\deg(K_C+\Delta_g^*) = 0$*), where Δ_g^* is the orbifold divisor on $\mathbb{P}^1$ which assigns to any $a \in \mathbb{P}^1$ the multiplicity 1 if $g(a) \notin$* $\mathrm{Supp}(\Delta)$*, and otherwise multiplicity $m_g(a) := \max_{j\in J(a)}\{\frac{m_j}{t_{j,a}}\}$, where $J(a), t_{j,a}$ are defined as in Definition 9.*

EXAMPLE 10. Assume now that $(X|\Delta)$ is of type $(m_0,\dots,m_{n+1})$ on $X=\mathbb{P}^n$, in the sense of example 5, and that C is a rational normal curve of degree n meeting each of the hyperplanes H_j in one point with contact order d, as in Theorem 8. Then C is virtually Δ-rational or elliptic, according to whether $(X|\Delta)$ is Fano, or has trivial canonical bundle (see Corollary 2 below). In general, C will not be Δ-rational, or elliptic, unless $n=2$. See Proposition 5 below.

COROLLARY 2. *Let $(\mathbb{P}^n|\Delta)$ be a smooth orbifold, the support of Δ consisting of the union of $(n+2)$ hyperplanes H_j in general position. The generic point of $\mathbb{P}^n$ is then contained in a normal rational curve C of degree n which meets each H_j in a single point, and such a curve is virtually Δ-rational (resp. Δ-elliptic) if $(\mathbb{P}^n|\Delta)$ is Fano (resp. has trivial canonical bundle).*

Moreover, if $n=2$, or $m_1\geq n$, or more generally if $\sum_{j=0}^{j=n+1}\frac{1}{m_j^*}>1$ (resp. if $\sum_{j=0}^{j=n+0}\frac{1}{m_j^*}=1$), then C is also Δ-rational (resp. Δ-elliptic), where $m_j^*:=\max\{m_j,n\}$.

Proof. The first assertion follows in the Fano case from Theorem 8 and the remark made before its statement. The same computation works in the trivial canonical bundle case: let m_j be the multiplicities of the H_j's. The orbifold $(\mathbb{P}^n|\Delta)$ has trivial canonical bundle if and only if: $\sum_j\frac{1}{m_j}=1$. The orbifold multiplicity at a point $a_j\in C$ is thus $\frac{m_j}{n}$. The corresponding orbifold divisor on C thus consists of $(n+2)$ points with multiplicities $\frac{m_j}{n}$, and this curve is thus Δ-elliptic if and only if $\sum_j\frac{n}{m_j}=(n+2)-2=n$, which holds true, since, by assumption, $\sum_j\frac{1}{m_j}=1$.

Let us now check the second assertion. If $m_1\geq n$, then $m_j\geq n,\forall j$, so that $\frac{m_j}{n}\geq 1,\forall j$, and so no max is needed to compute Δ_g in Definition 9. The conclusion thus follows. Notice that $m_1\geq 2$, so that the conclusion always holds when $n=2$.

Now if , for some $1\leq j\leq n+2$, $m_j<n$, taking $\max\{1,\frac{m_j}{t_{j,a}}=\frac{m_j}{n}\}$ amounts to replacing m_j by m_j^*, and so Δ_g^* is simply Δ_g computed for $(X|\Delta^*)$ instead of $(X|\Delta)$, with $\Delta^*:=\sum_{j=0}^{j=n+1}(1-\frac{1}{m_j^*}).H_j$, which implies the conclusion by the first part. □

We thus see that the consideration of rational normal curves of degree n permits us to show the uniruledness of some Fano smooth orbifolds of type $(m_0,\dots,m_{n+1})$, and of all if $n=2$. Unfortunately, when $n\geq 3$, these are, by far, not all Fano orbifolds of this type. We shall make now more precise which are the Fano smooth orbifolds of dimension $3=n$ which can be shown to be uniruled by this method, and which are not. First collecting the results of Theorem 8, Example 7, and Corollary 2, we get the assertions (1), (2), (3) below:

PROPOSITION 5. *Let $(\mathbb{P}^n|\Delta)$ be a smooth Fano orbifold of type $(m_0,\dots,m_{n+1})$. Then $(\mathbb{P}^n|\Delta)$ is uniruled, unless (maybe) if the following three conditions are realised:*

(1) $\sum\limits_j\frac{1}{m_j}>1$

(2) $\sum\limits_{j\neq(n)}\frac{1}{m_j}\leq 1$

(3) $\sum\limits_j \frac{1}{m_j^*} \leq 1$, *where* $m_j^* := \max\{m_j, n\}$, *if* $n = 3$.

(4) *If* $n = 3$, *then* $(\mathbb{P}^3|\Delta)$ *is uniruled, unless possibly if* $m_0 = 2, m_1 \geq 3$, *and* $m_2 \geq 4$.

Proof. We have only to prove the assertion (4). If $m_0 \geq 3$, then $m_j^* = m_j, \forall j$, and so (3) above contradicts (1). We thus assume that $m_0 = 2$ in the sequel. In the same way, if $m_1 = 2$, or if $m_1 = m_2 = 3$, then the sum of the first 2 or 3 terms of $\sum_j \frac{1}{m_j}$ is at least 1, contradicting (2). Thus $m_2 \geq 4$. □

When $n = 3$, the "Fano" types $(m_0, m_1, m_2, m_3, m_4)$ for which the uniruledness can (or cannot be) proved by the preceding method can be (lengthily) listed. We shall give in Example 12 two extremal cases when $n = 3$, for which the uniruledness $(\mathbb{P}^3|\Delta)$ cannot be proved by the preceding method.

DEFINITION 12. *Let* $(2 \leq a_1 \leq a_2 \leq \cdots \leq a_k)$ *be a finite sequence of positive integers such that* $\sum_{j=1}^{j=k} \frac{1}{a_j} = 1 - \frac{1}{b}$, *for some integer* $b \geq 2$. *Define inductively for* $s \geq 1$: $a_{k+1} := b + 1$, $a_{k+s+1} := a_{k+s}.(a_{k+s} + 1) + 1$. *It is then immediate to show inductively that* $\sum_{j=1}^{j=k+s} \frac{1}{a_j} = 1 - \frac{1}{a_{k+s+1} - 1}$ *for any* $s \geq 1$. *Thus*

$$\sum_{j=1}^{j=k+s} \frac{1}{a_j} + \frac{1}{a_{k+s+1} - 2} = 1 + \frac{1}{(a_{k+s+1} - 2).(a_{k+s+1} - 1)} > 1,$$

for any $s \geq 0$. *This will give us examples of Fano types on projective spaces.*

EXAMPLE 11. We look at special cases of the preceding sequences. They provide examples of types of Fano orbifolds which are uniruled, by Theorem 8 and its corollaries.

Let $k \geq 1$, and choose $a_1 = \cdots = a_k = k + 1$, so that $b = k + 1$. We then get $a_{k+1} = k + 2$, $a_{k+2} = (k+2)(k+3) + 1$, $a_{k+3} = a_{k+2}.(a_{k+2} + 1) + 1, \ldots$.
When $k = 1$, we get $d_1 = 2,\ d_2 = 3,\ d_3 = 7,\ d_4 = 6.7 + 1 = 43,\ d_4 = 42.43 + 1 = 1807, \ldots$ (setting $a_j = d_j$ in this case).
When $k = 2$, we get $t_1 = t_2 = 3$, $t_3 = 4$, $t_4 = 13$, $t_4 = 157, \ldots$ (setting $a_j = t_j$ in this case).
When $k = 4$, the sequence is: $q_1 = q_2 = q_3 = 4$, $q_4 = 5, q_5 = 21$, $q_6 = 421, \ldots$ (setting $a_j = q_j$ in this case).
When $k = n - 1$, the types $(m_1 = n, \ldots, m_{n-1} = n$, $m_n = (n+1)$, $m_{n+1} = n.(n+1) + 1$, $m_{n+2} = n.(n+1).(n^2 + n + 1) - 2$ are the types of Fano orbifolds $(\mathbb{P}^n|\Delta)$ with orbifold divisor supported on the union of $(n+2)$ hyperplanes in general position, and all such orbifolds are uniruled, by Theorem 8 and its corollaries.

Specific examples are thus: $n = 3$, and type $(3, 3, 4, 13, 155)$, or: $n = 4$ and type $(4, 4, 4, 5, 21, 419)$.

PROPOSITION 6 (See also [14]). *Let $N \geq 1$ be an integer. There exists a bound $B_N < 1$ such that if $2 \leq a_1 \leq \cdots \leq a_N$ is a finite sequence of integers, and if $A := \sum_{j=1}^{j=N} \frac{1}{a_j} < 1$, then $A \leq B_N$.*

Proof. We assume the existence of $B_N < 1$ and are going to establish inductively the existence of $B_{N+1} < 1$. It is plain that $B_1 = \frac{1}{2}$ exists. Assume now that $A + \frac{1}{a_{N+1}} := A' = \sum_{j=1}^{j=N+1} \frac{1}{a_j} < 1$. We can increase strictly A', preserving this last inequality, by replacing a_{N+1} by $a_{N+1} - 1$ (and possibly reordering the terms, in case $a_N = a_{N+1}$), unless $B_N + \frac{1}{a_{N+1}-1} \geq A + \frac{1}{a_{N+1}-1} \geq 1$. Thus, if A' cannot be increased in this way, as we may assume, we have $\frac{1}{a_{N+1}-1} \geq (1 - B_N)$, and $a_{N+1} \leq 1 + \frac{1}{1-B_N}$. There are thus only finitely many values for all a_j's, and there exists some $B_{N+1} < 1$ such that $A' \leq B_{N+1}$. □

EXAMPLE 12. The types $(m_0, m_1, m_2, m_3, m_4)$ for which $(\mathbb{P}^3|\Delta)$ is Fano, but which cannot be proved to be uniruled by the preceding method, satisfy in particular $m_0 = 2, 3 \leq m_1 \leq 7, 4 \leq m_2$ (this is easy, using 5). There are two main cases:

(a) $\sum_{0 \leq j \leq 3} \frac{1}{m_j} < 1$. There are only finitely many of them, and then $m_3 \leq d_4 = 43$, as may be shown using Proposition 6. Then also $m_4 < d_5 - 2$.

A typical example is $(2, 3, 7, 43, 1805)$.

(b) $\sum_{0 \leq j \leq 3} \frac{1}{m_j} \geq 1$. There are only finitely many such 4-tuples, since $\sum_{0 \leq j \leq 2} \frac{1}{m_j} < 1$, as follows from 5.3, and so $m_3 \leq 42 = d_4 - 1$. However, for any sufficiently large m_4, the given "type" satisfies the inequalities of 5.

Typical examples are $(2, 3, 7, 42, m_4)$, with any $m_4 \geq 42$.

REMARK 6. To show the uniruledness of Fano orbifolds of type $(2, 3, m_2, m_3, m_4)$ on $\mathbb{P}^3$, with $m_2 \geq 6$, it would be sufficient to show, for any 5-tuple of hyperplanes $H_j, j = 0, \ldots, 4$ of $\mathbb{P}^3$, the existence of a rational curve C of degree 6 tangent in 3 points to H_0, meeting H_1 in two distinct points with order of contact of order 3, and meeting each one of the three remaining H_j's in one single point with order of contact 6. Indeed the resulting C multiplicities would then be: 1 for the first 5 points, and $\frac{m_j}{6}$, for $j = 3, 4, 5$, and the last 3 points. This were a Δ-rational curve, since

$$\textstyle\sum_{2 \leq j \leq 4} \frac{6}{m_j} = 6.[(\sum_{0 \leq j \leq 4} \frac{6}{m_j}) - (\frac{1}{2} + \frac{1}{3})] > 6.(1 - \frac{5}{6}) = 1.$$

This construction appears to be similar to the one made above for the rational normal curves of degree 3. A simple dimension count shows that such curves should exist. The general case $n \geq 4$ seems to require other ideas and techniques, however.

8. Pseudoeffectivity of the relative canonical bundle

In this section, we shall prove by a relative Bend-and-Break technique, that the relative canonical bundle of a fibration is pseudoeffective if the generic fibre is not uniruled, and derive from it a weak version of Viehweg's weak positivity for the direct images of pluricanonical sheaves. Although the results are weaker than known ones, the method of proof is so straightforward, and possibly susceptible of further developments, that it seemed worth being written. Combined with Hodge-theoretic arguments, it might indeed permit one to easily obtain stronger versions, closer to Viehweg's results.

THEOREM 9. *Let $f : X \to Y$ be a fibration, with X, Y smooth projective connected. Assume that some smooth fibre X_y of f is not uniruled. Then $K_{X/Y}$ is pseudo-effective.*

Proof. Assume that $K_{X/Y}$ is not pseudo-effective. By [6], there exists an algebraic X-covering family $(C_t)_{t\in T}$ of curves such that $-K_{X/Y}.C_t > 0$. We can assume that the generic curve of this family is not rational, since it may be obtained as the direct image of a complete intersection of very ample divisors on some blow-up of X. Let $a \in X$ be a general point, lying on some smooth fibre X_y of f, and also on some nonrational irreducible member C of the family $(C_t)_{t\in T}$. Since $-K_{X/Y}.C > 0$, there exists, by [15, Proposition 3.11, p. 70], and the dimension estimate (2.4), p. 47, using the now standard Mori reductions to characteristic $p > 0$, a rational curve $R \subset X_y$ passing through a. Since X_y is not uniruled, choosing a not lying on any rational curve contained in X_y, we get a contradiction to our initial assumption that $K_{X/Y}$ is not pseudoeffective. □

Observe that the relative Bend-and-Break Lemma used above is the same as the one used in [9] and [22] to show the rational chain-connectedness of Fano manifolds.

We now deduce from the preceding Theorem 9 a proof of Proposition 3 when $\Delta = 0$.

COROLLARY 3. *Let $f : X \to Y$ be a fibration, with X, Y smooth, X projective, and let X_y be a generic fibre of f. Then*

(1) *Assume $\nu(X, K_X) = \kappa(X)$. Then $\kappa(X) \geq \kappa(X_y) + \kappa(Y)$.*

(2) *If X_y and Y are of general type, so is X.*

Proof. (1) We can assume that $\kappa(Y) \geq 0$. Let A be any ample divisor on X. Then $m.K_Y$, and also $m.K_{X/Y} + A$ are effective, for some $m \gg 0$, by Theorem 9 above. Thus $mK_X + A$, and so $N.K_X$ is effective, too, for some $N \gg 0$, since $\nu(X, K_X) = \kappa(X)$, by our assumption. The claim then follows from the arguments of [1, lemma 2.4, p. 516], for example.

(2) This follows from Lemma 4 below, applied to $P := K_{X/Y}$ and $D := K_Y$. □

LEMMA 4. *Let $f : X \to Y$ be a fibration. Let P be a pseudo-effective line bundle on X which is f-big (i.e., big on the generic fibre X_y of X). Then for any big $\mathbb{Q}$-divisor D on Y, $P + \varepsilon.f^*(D)$ is also big.*

Proof. $P + f^*(m.D)$ is big on X for some $m >> 0$, by the assumption of relative bigness. Since P is pseudo-effective, $(N-1)P + (P + m.f^*(D)) = N.(P + \frac{m}{N}.f^*(D))$ is also big, for any $N \geq 1$. Choosing $N \geq m$ establishes the claim. □

REMARK 7. (1) The proof of Theorem 9 does not seem to be able to give the weak positivity statement given by Viehweg's theorem. It also does not apply (directly at least) to the "orbifold" context of pairs $(X|\Delta)$. In this respect, it is much weaker.

(2) There is one point for which it is, however, more flexible: it does not need the effectivity of K_{X_y}, and its proof also directly gives information on multiples $m.K_{X/Y}$, contrary to Viehweg's proof which requires two steps: dealing first with $m = 1$, and then with arbitrary m's.

(3) Corollary 3 (2) is known in a much stronger version, by [21], which proves $C^+_{n,m}$ when X_y is of general type.

9. Families of canonically polarised manifolds

Roughly stated, a generalisation by Viehweg of a conjecture of Shafarevich states that "the moduli space of canonically polarised manifolds has components of log-general type". The initial formulation was that if $f : X \to B$ is an algebraic smooth family of canonically polarised manifolds parametrised by a quasi-projective manifold B having generically a "variation" (i.e., a Kodaira–Spencer map) of maximal rank, then B is of log-general type, considering a smooth compactification $B = Y - D$, such that $(Y|D)$ is smooth, with D reduced and of snc.

This has been shown in low dimension and various formulations by Viehweg–Zuo, Kebekus–Kovacs, Jabbusch–Kebekus (see [30] and [17] for the appropriate references).

The natural formulation of this conjecture seems to be:

CONJECTURE 10 (The "Isotriviality Conjecture"). Let $f : X \to B$ be as above, assume that B is special (i.e., that so is any smooth compactification $(Y|D)$ as above. Then f is isotrivial (i.e., all fibres of f are isomorphic).

This implies Viehweg's Conjecture, since the moduli map (for arbitrary families $f : X \to B$) then factors through the core of $(Y|D)$, which is of log-general type.

THEOREM 10 ([8]). *The Isotriviality Conjecture follows from Conjectures* $C^{orb}_{n,m}$ *and 7.*

REMARK 8. The Isotriviality Conjecture is thus reduced to standard conjectures of birational geometry[5].

Sketch of proof. The proof rests essentially on the construction by Viehweg–Zuo of a line bundle $L \subset \mathrm{Sym}^m(\Omega^1_Y(\log D))$ such that $\kappa(Y,L) = \mathrm{Var}(f)$ (see [30]). Because, as a consequence of $C^{orb}_{n,m}$ and Theorem 6 we have a canonical decomposition $c = (J \circ r^*)^n$ which is the constant map (since $B = Y - D$ is special), it is sufficient to show

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[5] It was stated in [8] that the Isotriviality Conjecture follows from Conjecture 3. But it is only true that it follows from the arguments used to deduce Conjecture 3 from Conjectures $C^{orb}_{n,m}$ and 7. The proof given in the present text simplifies the arguments given in [8].

the result when either $\kappa(B)=0$, or $\kappa_+(B)=-\infty$. In the first (resp. second) case, we have, by Conjecture 7 (1), the existence of a sequence of divisorial contractions and flips $s:(X|\Delta)\dashrightarrow(X'|\Delta')$ such that $c_1(X'|\Delta')=0$ (resp. such that a non-trivial fibration $f:(Y|D)\to Z$ with Fano orbifold fibres $(Y'_z|\Delta'_z)$ exists). In the first case, we directly conclude that $\kappa(Y,L)\leq 0$, which implies that $\mathrm{Var}(f)=0$, as claimed. In the second case, we are reduced, by the equality $r^*=r^n$ of Theorem 3 to the case where $(Y|D)$ is Fano. Considering, as in the proof of Lemma 2, a new orbifold divisor $\Delta^+=D+\frac{1}{m.n}.H>D$, with $H\in|-m.n.(K_Y+D)|$, so that $c_1(Y|\Delta^+)=0$, we conclude $\kappa(Y|D,L)=-\infty$, since $\kappa(Y|\Delta^+)\leq 0$, so that the family is isotrivial on these fibres. □

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THE FIXED POINT SET OF ANTI-SYMPLECTIC INVOLUTIONS OF LAGRANGIAN FIBRATIONS

Abstract. We discuss some results and ideas on the topology of Lagrangian submanifolds obtained as the fixed point locus of certain anti-symplectic involutions preserving the fibres of a Lagrangian fibration $f: X \to B$. Here X is a symplectic manifold diffeomorphic to a Calabi–Yau manifold.

1. Introduction

The fixed point locus of an anti-symplectic involution (i.e. a map $\iota: X \to X$ such that $\iota^2 = \mathrm{Id}_X$ and $\iota^*\omega = -\omega$) is an interesting type of Lagrangian submanifold of a symplectic manifold (X, ω). An easy, classical, construction of such an involution is given by complex conjugation when X is a smooth complex subvariety of $\mathbb{P}^n$ cut out by polynomials with real coefficients. In this case the fixed point locus is just the intersection with $\mathbb{RP}^n$. Understanding the topology of such varieties is generally a difficult problem. One reason why the fixed point set of an anti-symplectic involution is interesting is that its Floer homology is particularly well behaved ([5, 16]). In [3], together with Jake P. Solomon, we constructed a class of anti-symplectic involutions by requiring that they preserve the fibres of the Lagrangian fibrations $f: X \to B$ constructed in [2]. In this case X is diffeomorphic to a Calabi–Yau manifold (of complex dimension 2 or 3), e.g. a $K3$ surface or a quintic hypersurface in $\mathbb{P}^4$.

In this note we review the constructions in [2] and [3] and we report, in an informal way and with almost no proofs, on some work in progress on the topology of the fixed point locus of these anti-symplectic involutions. Proofs and details, together with other results, will appear in [1]. Many of the results and ideas mentioned here on Lagrangian fibrations are based on, or inspired by, the work of M. Gross [6, 7, 8, 10] and M. Gross–B. Siebert [12]. In particular, it follows from results in these articles and the construction in [2], that in most cases also the mirror Calabi–Yau $\check{X}$ comes with a “dual” Lagrangian fibration and anti-symplectic involution. In our fibrations the general fibre of f is a smooth Lagrangian torus, while fibres over points in a set $\Delta \subset B$ are singular. In the 2-dimensional case the fibrations we consider are topologically identical to stable elliptic fibrations, i.e. $B = S^2$ and there are 24 singular fibres of Kodaira type I_1, i.e. once pinched tori. In the 3-dimensional case the base is homeomorphic to S^3 and the discriminant locus is a 3-valent graph (with the possibility that some connected components are just circles with no vertices). The singular fibres are also “stable” in some sense but their topology is more complicated.

The fibrations we consider also have a Lagrangian section, therefore the smooth fibres have naturally a group structure isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. The anti-symplectic involution fixes such a section and restricted to smooth fibres is just $\alpha \mapsto -\alpha$, and therefore the fixed point set in each smooth fibre is just 2^n points. So, if we call Σ the fixed point locus, then Σ is a Lagrangian submanifold of X and f restricted to Σ is a branched covering of B, of degree 2^n, branching over Δ. In the 2-dimensional case it is not difficult to show that Σ has two connected components, one being the fixed Lagrangian section, the other being a genus 10 curve. In this case Σ has the largest possible total cohomology group for the fixed point locus of an involution, and is therefore maximal. The 3-dimensional case is more complicated. We discuss a long exact sequence linking the $\mathbb{Z}/2\mathbb{Z}$ cohomology of Σ to the cohomology of X. It is inspired by a Leray spectral sequence studied by Gross (op. cit.), computing the cohomology of X in terms of the fibration. As a corollary we obtain that if X and its mirror $\check{X}$ are simply connected, then Σ has just two connected components. Computing the $\mathbb{Z}/2\mathbb{Z}$-cohomology of Σ is reduced to computing a map β appearing in the long exact sequence (cf. Section 3). When $\beta = 0$, Σ has the largest possible total cohomology. We describe an explicit example coming from the so-called Schoen's Calabi–Yau, where Σ can be described in a sufficiently simple way so to apply standard techniques for the computation of cohomology. The result is that for Schoen's 3-fold, β is not zero.

A few questions remain open. Can we compute β explicitly in more complicated known examples such as the quintic or complete intersections in toric manifolds? What is the relationship between Σ and the corresponding fixed point locus $\check{\Sigma}$ inside the mirror manifold $\check{X}$? What is the relation between the involutions we study and the more classical ones constructed algebraically, for instance by conjugation in $\mathbb{P}^n$? Is Σ in our case somehow special among other possible constructions, i.e. is it maximal in some other sense? These and other questions will be addressed in [1] and further work.

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2. Lagrangian fibrations and involutions

2.1. Affine manifolds with singularities

Let (X,ω) be a smooth symplectic $2n$-dimensional manifold, B a smooth n-dimensional manifold and $\Delta \subset B$ a closed subset with $B_0 = B - \Delta$ dense in B. A *Lagrangian fibration* on X is a smooth map $f : X \to B$ such that the fibres of f over B_0 are Lagrangian submanifolds (i.e. dim $f^{-1}(b) = n$ and $\omega|_{f^{-1}(b)} = 0$), and f restricted to B_0 is a submersion. The fibres over Δ are called singular. If the top dimensional stratum of a singular fibre is n-dimensional, we require the smooth part of it, $f^{-1}(b) - \{\mathrm{Crit}(f) \cap f^{-1}(b)\}$, to be a Lagrangian submanifold as well. When the fibres are compact and connected, then the Arnold-Liouville theorem implies that the fibres over B_0 are all n-tori. Moreover, we can cover the subset B_0 with an atlas, $\{(U_j,\phi_j)\}_{j\in J}$ such that the transition maps are affine transformations whose linear part has integral coefficients, i.e. $\phi_j \circ \phi_k^{-1} \in \mathbb{R}^n \rtimes \mathrm{Sl}_n(\mathbb{Z})$.

This motivates the definition of an *integral affine manifold with singularities*: a topological manifold B with a closed subset $\Delta \subset B$, with $B_0 = B - \Delta$ dense in B, such that on B_0 there exists an atlas $\mathcal{A}$ whose change of coordinates are in $\mathbb{R}^n \rtimes \mathrm{Sl}_n(\mathbb{Z})$. If $(y_1,\ldots,y_n)$ are affine coordinates, then the $\mathbb{Z}$-linear combinations of the 1-forms $dy_1,\ldots dy_n$ span a maximal lattice $\Lambda \subset T^*B_0$ which is well defined independently of the chosen affine coordinates (this follows from the fact that the linear part of the change of coordinates is in $\mathrm{Sl}_n(\mathbb{Z})$). We can use this to form the n-torus bundle $X_0 = T^*B_0/\Lambda$. The standard symplectic form on T^*B_0 descends to X_0 so that the standard projection $f_0 : X_0 \to B_0$ is a Lagrangian submersion.

In general, if we start with a given integral affine manifold with singularities $(B,\Delta,\mathcal{A})$, we may ask whether we can find a symplectic manifold X and extend the bundle $f_0 : X_0 \to B_0$ to a Lagrangian fibration $f : X \to B$ by inserting singular Lagrangian fibres over the set Δ. More precisely we want the following commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "j"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ B_0 \arrow[r, "i"] & B \end{tikzcd} \tag{1}$$

where j is a symplectomorphism and i is the inclusion. This is the starting point for the construction of the Lagrangian fibrations in [2]. If we ask the question at the purely topological level (i.e. without requiring a symplectic form on X and the Lagrangian condition on f) then, for the cases we consider here, the answer was provided by Gross in [8]. In particular Gross finds a topological torus fibration on the quintic threefold in $\mathbb{P}^4$.

Let us now give some examples of affine manifolds with singularities.

EXAMPLE 1 (Focus-focus). We start with a 2-dimensional example. We define an affine structure with singularities on $B = \mathbb{R}^2$. Let $\Delta = \{0\}$ and let (x_1,x_2) be the standard coordinates on B. As the covering $\{U_i\}$ of $B_0 = \mathbb{R}^2 - \Delta$ we take the following two sets

$$U_1 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \geq 0\},$$

$$U_2 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \leq 0\}.$$

Denote by H^+ the set $\{x_2 > 0\}$ and by H^- the set $\{x_2 < 0\}$. Let T be the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2}$$

The coordinate maps ϕ_1 and ϕ_2 on U_1 and U_2 are defined as follows

$$\phi_1 = \mathrm{Id},$$

$$\phi_2 = \begin{cases} \mathrm{Id} & \text{on } \bar{H}^+ \cap U_2 \\ (T^{-1})^t & \text{on } H^-. \end{cases}$$

The atlas $\mathcal{A} = \{U_i, \phi_i\}_{i=1,2}$ is clearly an affine structure on B_0. We can easily check that if we consider the 2-torus bundle $X_0 = T^*B_0/\Lambda$, then the monodromy of the H_1 homology of the fibres at a point $b \in B_0$, associates the matrix T to the anti-clockwise oriented generator of $\pi_1(B_0)$.

This is a compact example:

EXAMPLE 2. In $\mathbb{R}^3$ consider the 3-dimensional simplex Ξ spanned by the points $P_0 = (-1,-1,-1)$, $P_1 = (3,-1,-1)$, $P_2 = (-1,3,-1)$, $P_3 = (-1,-1,3)$. Let $B = \partial\Xi$. We explain how to construct an affine structure with singularities on B. Each edge ℓ_j of Ξ has 5 integral points (i.e. belonging to $\mathbb{Z}^3$), which divide ℓ_j into 4 segments. For each $j = 1,\dots,6$ denote by Δ^j_k, $k = 1,\dots,4$ the four barycenters of these four segments. We let

$$\Delta = \{\Delta^j_k;\, j = 1\dots 6 \text{ and } k = 1,\dots,4\}.$$

A covering of $B_0 = B - \Delta$ can be defined as follows. The first four open sets consist of the four open faces Σ_i, $i = 1\dots,4$ with the affine coordinate maps ϕ_i induced by their affine embeddings in $\mathbb{R}^3$. Denote by I the set of integral points of B which lie on an edge. For every $Q \in I$ we can choose a small open set U_Q in B_0 such that $\{\Sigma_i\}_{i=1,\dots,4} \cup \{U_Q\}_{Q\in I}$ is a covering of B_0. Let R_Q denote the 1-dimensional subspace of $\mathbb{R}^3$ generated by $Q \in I$. One can verify that if U_Q is small enough, the projection $\phi_Q : U_Q \to \mathbb{R}^3/R_Q$ is a homeomorphism. A computation shows that the atlas $\mathcal{A} = \{\Sigma_i, \phi_i\}_{i=1,\dots,4} \cup \{U_Q, \phi_Q\}_{Q\in I}$ defines an integral affine structure on B_0.

In the latter example it can be easily checked that a neighbourhood of the singular points in Δ is affine isomorphic to a neighbourhood of $0 \in \mathbb{R}^2$ in Example 1. In dimension 2, an affine manifold with singularities $(B, \Delta, \mathcal{A})$ is called *simple* if Δ consists of isolated points and each point has a neighbourhood affine isomorphic to a neighbourhood of 0 in Example 1.

We now present some 3-dimensional examples.

EXAMPLE 3 (The edge). Let $I \subseteq \mathbb{R}$ be an open interval. Consider $B = \mathbb{R}^2 \times I$ and $\Delta = \{0\} \times I$. On $B_0 = (\mathbb{R}^2 - \{0\}) \times I$ we take the product affine structure between the affine structure on $\mathbb{R}^2 - \{0\}$ described in Example 1 and the standard affine structure on I.

EXAMPLE 4 (Positive vertex). Let $B = \mathbb{R} \times \mathbb{R}^2$ and let (x_1, x_2, x_3) be coordinates in B. Identify $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$. Inside $\mathbb{R}^2$ consider the cone over three points:

$$\Delta = \{x_2 = 0, x_3 \le 0\} \cup \{x_3 = 0, x_2 \le 0\} \cup \{x_2 = x_3, x_3 \ge 0\}.$$

Now define closed sets in B

$$\begin{aligned} R &= \mathbb{R} \times \Delta, \\ R^+ &= \mathbb{R}_{\ge 0} \times \Delta, \\ R^- &= \mathbb{R}_{\le 0} \times \Delta, \end{aligned}$$

and consider the following cover $\{U_i\}$ of $\mathbb{R}^3 - \Delta$:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - R^+, \\ U_2 &= \mathbb{R}^3 - R^-. \end{aligned}$$

It is clear that $U_1 \cap U_2$ has the following three connected components

$$\begin{aligned} V_1 &= \{x_2 < 0,\ x_3 < 0\}, \\ V_2 &= \{x_2 > 0,\ x_2 > x_3\}, \\ V_3 &= \{x_3 > 0,\ x_3 > x_2\}. \end{aligned}$$

Take two matrices

$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Now on U_1, U_2 we define coordinate maps ϕ_1, ϕ_2 as follows

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} \mathrm{Id} & \text{on } \bar{V}_1 \cap U_2 \\ T_1^{-1} & \text{on } \bar{V}_2 \cap U_2 \\ T_2 & \text{on } \bar{V}_3 \cap U_2. \end{cases} \end{aligned}$$

Again we see that $\mathcal{A} = \{U_i, \phi_i\}_{i=1,2}$ gives an affine structure on $B_0 = \mathbb{R}^3 - \Delta$. One can compute that, if we form the 3-torus bundle X_0, given a point $b \in B_0$ and two closed paths generating $\pi_1(B_0)$, then the H_1 monodromy of the fibre of X_0 associates to these two paths the matrices $(T_j^{-1})^t$ for $j = 1, 2$.

EXAMPLE 5 (Negative vertex). Let B and Δ be as in Example 4. Then, $\mathbb{R}^2 - \Delta$ has three connected components, which we denote C_1, C_2 and C_3. Let $\bar{C}_j = C_j \cup \partial C_j$. Consider the following three open subsets of B_0:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - (\bar{C}_2 \cup \bar{C}_3), \\ U_2 &= \mathbb{R}^3 - (\bar{C}_1 \cup \bar{C}_3), \\ U_3 &= \mathbb{R}^3 - (\bar{C}_1 \cup \bar{C}_2). \end{aligned}$$

Let

$$\begin{aligned} V^+ &= \{x_1 > 0\}, \\ V^- &= \{x_1 < 0\}. \end{aligned}$$

Clearly $U_i \cap U_j = V^+ \cup V^-$ when $i \neq j$. If T_1 and T_2 are as in (3), define the following coordinate charts on U_1, U_2, U_3 respectively:

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} (T_1^{-1})^t & \text{on } \bar{V}^+ \cap U_2 \\ \mathrm{Id} & \text{on } \bar{V}^- \cap U_2, \end{cases} \\ \phi_3 &= \begin{cases} \mathrm{Id} & \text{on } \bar{V}^+ \cap U_3 \\ (T_2^{-1})^t & \text{on } \bar{V}^- \cap U_3. \end{cases} \end{aligned}$$

We can check that the affine structure defined by these charts is such that, on the 3-torus bundle X_0, given a point $b \in B_0$, then the H_1 monodromy of the fibre of X_0 associates to two generators of $\pi_1(B_0)$ the matrices $T_j,\ j = 1,2$. In particular, monodromy is given by the inverse transpose matrices of the monodromy in the previous example.

These three examples are the building blocks of so-called 3-dimensional simple affine structures with singularities. A 3-dimensional compact example is the following:

EXAMPLE 6. This three dimensional example is taken from [11, §19.3]. Let Ξ be the 4-simplex in $\mathbb{R}^4$ spanned by

$$P_0 = (-1,-1,-1,-1),\ P_1 = (4,-1,-1,-1),\ P_2 = (-1,4,-1,-1),$$
$$P_3 = (-1,-1,4,-1),\ \ P_4 = (-1,-1,-1,4).$$

Let $B = \partial\Xi$. Denote by Σ_j the open 3-face of B opposite to the point P_j and by F_{ij} the closed 2-face separating Σ_i and Σ_j. Each F_{ij} contains 21 integral points (including those on its boundary). These form the vertices of a triangulation of F_{ij} as in Figure 1. By joining the barycenter of each triangle with the barycenters of its sides we form a trivalent graph as in Figure 1. Define the set Δ to be the union of all such graphs in each 2-face. Denote by I the set of integral points of B. Just as in the previous example, we can form a covering of $B_0 = B - \Delta$ by taking the open 3-faces Σ_j and small open neighborhoods U_Q inside B_0 of $Q \in I$. A coordinate chart ϕ_i on Σ_i can be obtained from its affine embedding in $\mathbb{R}^4$. If we denote again by R_Q the linear space spanned by $Q \in I$, as a chart on U_Q we take the projection $\phi_Q : U_Q \to \mathbb{R}^4/R_Q$.

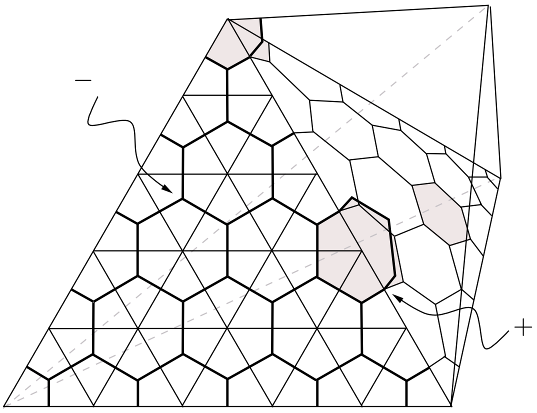

Figure 1: Affine S^3 with singularities.

In the above example one can check that points in the interior of the edges of the graph Δ have neighbourhoods which are affine isomorphic to neighbourhoods of points on the line of singularities in Example 3. The vertices of Δ which are in the interior of 2-faces have neighbourhoods affine isomorphic to the vertex in Example 5 (these are called negative vertices), while vertices of Δ on 1-faces have neighbourhoods affine isomorphic to a neighbourhood of the vertex in Example 4 (positive vertices).

We say that a 3-dimensional affine manifold with singularities is *simple* if Δ is a 3-valent graph, with vertices labelled as positive or negative. The affine structure near points on the edges of Δ is locally affine isomorphic to Example 3, near positive (resp. negative) vertices it is locally affine isomorphic to Example 4 (resp. Example 5).

2.2. Glueing singular fibres

Given the symplectic manifold $X_0 = T^*B_0/\Lambda$, how do we glue singular fibres to X_0? The 2-dimensional case can be achieved as follows. First consider the following example of Lagrangian fibration:

EXAMPLE 7. Let $X = \mathbb{C}^2 - \{z_1z_2 + 1 = 0\}$ and let ω be the restriction to X of the standard symplectic form on $\mathbb{C}^2$. One can easily check that the following map $f : X \to \mathbb{R}^2$ is a Lagrangian fibration:

$$f(z_1,z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log|z_1z_2 + 1| \right). \tag{4}$$

The only singular fibre is $f^{-1}(0)$, which has the topology of a I_1 fibre (a pinched torus).

This is an example of a fibration of "focus-focus" type. One can explicitly compute the affine coordinates on the base, away from the singular point $(0,0) \in \mathbb{R}^2$. It can be shown that this affine structure is isomorphic (in a neighbourhood of $(0,0)$) to the one given in Example 1. This implies that given a 2-dimensional, simple, affine manifold with singularities and a point $p \in \Delta$, we can glue, via a fibre-preserving symplectomorphism, a neighbourhood of the singular fibre in the above example to $(f_0)^{-1}(U - p) \subset X_0$ for a suitable neighbourhood U. For the details of this construction consult [2]. If we do this at all 24 points in Example 2, in the end we obtain a symplectic manifold diffeomorphic to a K3 surface and a Lagrangian fibration $f : X \to S^2$ with 24 singular fibres and a Lagrangian section of f.

A similar, but rather more complicated, construction can be carried out in the case of a 3-dimensional, simple affine manifold with singularities. Thus obtaining a 6-dimensional (compact) symplectic manifold X with a Lagrangian fibration $f : X \to B$, together with a Lagrangian section. This is the main result of [2]. The idea is to find suitable models of Lagrangian fibrations with singular fibres which can be glued over Δ. When compactified in this way, Example 6 gives a manifold diffeomorphic to a quintic in $\mathbb{P}^4$. We should warn the reader that in the final result of [2] the map f is not smooth but just piecewise smooth, it fails to be smooth only along the preimage of small 2-dimensional discs containing negative vertices. Also, the discriminant locus Δ has to be enlarged slightly, so that near a negative vertex it is a codimension 1 thickening of the graph. The total space X obtained is nevertheless smooth. When the integral affine base is as the ones considered by Gross and Siebert, X turns out to be diffeomorphic to a Calabi–Yau.

2.3. Anti-symplectic involutions

An *anti-symplectic involution* on a symplectic manifold (X,ω) is a map $\iota : X \to X$ such that $\iota^*\omega = -\omega$ and $\iota^2 = \mathrm{Id}_X$. The fixed point set of an anti-symplectic involution is always a Lagrangian submanifold. In [3], together with Jake P. Solomon, we showed that, given a Lagrangian fibration $f : X \to B$ with a Lagrangian section constructed as above, one can also find an anti-symplectic involution $\iota : X \to X$ which preserves the fibres and fixes the section. The idea for the construction is as follows. Consider the fibre-preserving anti-symplectic involution ι_0 on T^*B_0/Λ induced by $(p,\alpha) \mapsto (p,-\alpha)$ for every $p \in B_0$ and $\alpha \in T^*_pB_0$. We show that ι_0 extends to a smooth fibre-preserving anti-symplectic involution ι on X. This is done by first studying anti-symplectic involutions on local models of singular fibres and then refining the gluing by also matching the involutions.

In this note, we would like to discuss the topology of the fixed point set of this type of involutions. The fixed point set Σ is the closure in X of the image of the set $\frac{1}{2}\Lambda$ inside T^*B_0/Λ. The map $f|_\Sigma : \Sigma \to B$ is a branched covering of B, branching over Δ. In dimension 2 this branched covering is of degree 4 and of degree 8 in dimension 3. Let us look more closely at some examples.

EXAMPLE 8. In the “focus-focus” case, i.e. Examples 1 and 7, fix a point $b \in \mathbb{R}^2 - \{(0,0)\}$. Then we can find a basis of Λ_b with respect to which, monodromy is the matrix (2). With respect to this basis, we can identify Λ_b with $\mathbb{Z}^2$. Then $\Sigma \cap T^*_bB_0/\Lambda_b$ consists of points $s_0 = (0,0), s_1 = (1/2,0), s_2 = (0,1/2)$ and $s_3 = (1/2,1/2)$. As we go around the singular point, monodromy maps s_0 and s_2 to themselves and s_1 to s_3. Therefore, if U is a neighbourhood of $(0,0)$, $f^{-1}(U) \cap \Sigma$ has 3 connected components. There are two which map 1 to 1 to U, these are the ones containing s_0 and s_2 respectively. Then there is one mapping 2 to 1, which contains both s_1 and s_3. The map f restricted to this latter component is a 2 to 1 branched covering, with branched point inside the singular fibre.

EXAMPLE 9. In the case of Example 2, where the compactified X is diffeomorphic to a K3 surface, the fixed point set Σ is a compact Lagrangian surface. It is not difficult to check that Σ has 2 connected components, one of them is the zero section and the other one is a degree 3 branched covering over S^2 with 24 branched points of ramification index 2. The Riemann–Hurwitz formula tells us that this connected component is a genus 10 surface.

Observe that in the previous example $b^1(\Sigma) = 20 = h^{1,1}(X)$. We will discuss in the following sections the reason why this equality is not a coincidence. Moreover it is known that the fixed point set Σ of an involution $\iota : X \to X$ on a compact manifold satisfies the following inequality in cohomology (see [4]):

$$\dim\left(\sum_{*} H^{*}(\Sigma,\mathbb{Z}/2\mathbb{Z})\right) \leq \dim\left(\sum_{*} H^{*}(X,\mathbb{Z}/2\mathbb{Z})\right) \tag{5}$$

In the case of our involution, Σ satisfies the equality, so it is in some sense maximal.

2.4. Mirror symmetry

For a more thorough explanation of the relevance of affine manifolds with singularities and Lagrangian torus fibrations in the context of mirror symmetry the reader may consult [9] and the references therein. We just mention a few facts that are related to the topic of this note. Given an affine manifold with singularities B, besides $X_0 = T^*B_0/\Lambda$, we can also construct $\check{X}_0 = TB_0/\Lambda^*$, where Λ^* is the dual lattice. If B is simple, then also $\check{X}_0$ can be topologically compactified. This follows from the Gross's topological compactification, in fact in this case the role of vertices is inverted: where we had negative vertices, we glue fibres of positive type and viceversa. We thus obtain a (compact) smooth manifold $\check{X}$ together with a torus fibration $\check{f}: \check{X} \to B$. It was shown by Gross (see next subsection) that the manifolds X and $\check{X}$ satisfy the topological conditions required for them to be mirror manifolds. In the case of Example 6, Gross also shows that $\check{X}$ is diffeomorphic to the mirror of the quintic.

The existence of a good symplectic structure on $\check{X}$ is not immediately apparent from this description, since the tangent bundle does not carry a natural symplectic structure. To solve this problem one needs the extra data on B of a (multivalued) strictly convex function ϕ, which can be used to define a symplectic form on TB_0. Equivalently, via the Legendre transform applied to ϕ, one defines a new affine structure on B_0, giving a new lattice in T^*B_0, which we denote by $\check{\Lambda}$. It can be checked that $T^*B_0/\check{\Lambda}$ and $\check{X}_0$ are isomorphic torus bundles over B_0, therefore also $\check{X}_0$ has a symplectic structure (inherited from T^*B_0). Thus, also $\check{X}_0$ can be symplectically compactified to give a Lagrangian fibration $\check{f}: \check{X} \to B$, with a fibre-preserving anti-symplectic involution. Using ideas from toric geometry, Gross and Siebert also introduce the discrete Legendre transform, which is a combinatorial version of the standard Legendre transform. In [10], Gross shows that this construction can be applied to all the examples of Batirev–Borisov's pairs of mirror Calabi–Yau's.

In [3] we also discuss the relevance of our construction of anti-symplectic involutions in the context of the Homological Mirror Symmetry conjecture. This conjecture states that given mirror manifolds X and $\check{X}$, there should be an equivalence of categories between the derived category of coherent sheaves on $\check{X}$ and the derived Fukaya category on X. The objects in this latter category are Lagrangian submanifolds of X, with some other data attached. Since the two categories are conjectured to be equivalent, an autoequivalence on one category should correspond to one on the other. The category of coherent sheaves has a natural autoequivalence which consists in mapping a sheaf to its dual. In [3] we discussed some evidence of a conjecture claiming that the autoequivalence on the Fukaya category, corresponding to dualization, should be given by the anti-symplectic involution ι that we constructed, where a Lagrangian submanifold is mapped to its image under ι.

2.5. Topology of Lagrangian 3-torus fibrations

We describe here some of Gross's results on the Leray spectral sequence applied to the torus fibrations $f : X \to B$ of the type discussed in the previous section. We assume that $(B,\Delta,\mathcal{A})$ is a compact simply-connected, 3-dimensional, simple integral affine manifold with singularities. The arguments are entirely topological, without any reference to the fact that X is symplectic and the fibres are Lagrangian. Given an abelian group G, we denote by $R^k f_* G$ the sheaf associated to the presheaf on B given by $U \mapsto H^k(f^{-1}(U), G)$. The Leray spectral sequence associated to f has as E_2 terms the groups $H^j(B, R^k f_* G)$. We recall Gross's definition:

DEFINITION 1. *Let $i : B_0 \to B$ be the inclusion. The fibration $f : X \to B$ is G-simple if*

$$i_* R^k f_{0*} G = R^k f_* G$$

We will assume in the following that X is simply connected. For some of the arguments, this condition can be relaxed, e.g. it could be replaced with $H^1(X,\mathbb{R}) = 0$. In [8] Gross showed that the fibrations considered here are always G-simple, when G is $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. Moreover $\mathbb{Z}$ simplicity implies $\mathbb{Q}$ simplicity. Notice also that, since affine coordinates on B_0 have linear part in $\mathrm{Sl}(n,\mathbb{Z})$, the fibres are canonically oriented, so that

$$R^3 f_* G = G.$$

Moreover, f has a smooth section (extending the zero section on T^*B_0). We also consider the mirror dual fibration $\check{f} : \check{X} \to B$, which is also a G-simple fibration with a section and we will assume that also $\check{X}$ is simply connected.

Now, let $G = \mathbb{Q}$ (but the following also holds for $G = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$). By Poincaré duality applied to the fibres, we have that

$$(R^j f_{0*}\mathbb{Q})^\vee = R^{n-j} f_{0*}\mathbb{Q}.$$

Moreover, from the definition of dual torus fibration, we also have:

$$(R^j f_{0*}\mathbb{Q})^\vee = R^j \check{f}_{0*}\mathbb{Q}.$$

By applying i_* to the above and using $\mathbb{Q}$-simplicity we obtain

$$R^j \check{f}_*\mathbb{Q} = R^{n-j} f_*\mathbb{Q}. \tag{6}$$

The E_2 page for the Leray spectral sequence for f with $G = \mathbb{Q}$, looks like the following

$$\begin{array}{cccc}
\mathbb{Q} & 0 & 0 & \mathbb{Q} \\
0 & H^1(B, R^2 f_*\mathbb{Q}) & H^2(B, R^2 f_*\mathbb{Q}) & 0 \\
0 & H^1(B, R^1 f_*\mathbb{Q}) & H^2(B, R^1 f_*\mathbb{Q}) & 0 \\
\mathbb{Q} & 0 & 0 & \mathbb{Q}
\end{array}$$

For the proof of this, one can argue as follows. Since the fibres are connected, we have $R^0 f_*\mathbb{Q} = \mathbb{Q}$. Moreover $R^3 f_*\mathbb{Q} = \mathbb{Q}$, as we already mentioned. So, together with the fact that B is simply connected, we obtain the zeroes in the top and bottom row. The zeroes in the first and last column come from the fact that $H^1(X,\mathbb{Q}) = H^5(X,\mathbb{Q}) = H^1(\check{X},\mathbb{Q}) = H^5(\check{X},\mathbb{Q}) = 0$ together with (6). The E_2 term for $\check{f}$ is obtained by exchanging the first and second row of the E_2 term of f.

Gross proved that under these hypotheses the Leray spectral sequences of f and $\check{f}$ degenerate at the E_2 term, so that when X is a Calabi–Yau manifold

$$h^{1,1}(X) = \dim H^1(B, R^1 f_*\mathbb{Q}) = \dim H^1(B, R^2 \check{f}_*\mathbb{Q}) = h^{2,1}(\check{X}).$$

So the topology of these fibrations on X and $\check{X}$ guarantees that X and $\check{X}$ satisfy the basic topological requirement of mirror symmetry. These arguments also work if we replace $\mathbb{Q}$ with $\mathbb{Z}/2\mathbb{Z}$ (except, maybe, the equality with Hodge numbers, due to possible presence of 2-torsion):

$$\begin{array}{cccc}
\mathbb{Z}/2\mathbb{Z} & 0 & 0 & \mathbb{Z}/2\mathbb{Z} \\
0 & H^1(B, R^2 f_*\mathbb{Z}/2\mathbb{Z}) & H^2(B, R^2 f_*\mathbb{Z}/2\mathbb{Z}) & 0 \\
0 & H^1(B, R^1 f_*\mathbb{Z}/2\mathbb{Z}) & H^2(B, R^1 f_*\mathbb{Z}/2\mathbb{Z}) & 0 \\
\mathbb{Z}/2\mathbb{Z} & 0 & 0 & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

From which we obtain that

$$H^2(X,\mathbb{Z}/2\mathbb{Z}) \cong H^1(B, R^1 f_*\mathbb{Z}/2\mathbb{Z})$$

$$H^3(X,\mathbb{Z}/2\mathbb{Z}) \cong H^1(B, R^2 f_*\mathbb{Z}/2\mathbb{Z}) \oplus H^2(B, R^1 f_*\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$H^4(X,\mathbb{Z}/2\mathbb{Z}) \cong H^2(B, R^2 f_*\mathbb{Z}/2\mathbb{Z})$$

3. A long exact sequence

We now wish to understand the $\mathbb{Z}_2$-cohomology of the fixed point locus of the anti-symplectic involutions constructed in the previous section. We sketch here the construction of a long exact sequence which links the cohomology of the ambient manifold X with the cohomology of Σ, details will appear in [1]. The assumptions on $f: X \to B$ are the same as those in the last subsection of the previous section (in particular X is 6-dimensional) and we let Σ be the fixed point locus of the anti-symplectic involution $\iota: X \to X$. Let

$$\sigma = f|_{\Sigma}$$

and denote by σ_0 the restriction of σ to $\sigma^{-1}(B_0)$. The idea is to consider the spectral sequence associated to the branched covering $\sigma: \Sigma \to B$ and compare it with the one associated to f. Observe that the E_2 term of the spectral sequence of σ consists of just one row of elements of the type $H^j(B, R^0\sigma_*\mathbb{Z}/2\mathbb{Z})$ and therefore the spectral sequence degenerates at E_2. It can also be shown that, since f is $\mathbb{Z}/2\mathbb{Z}$-simple then also $\sigma: \Sigma \to B$ is a $\mathbb{Z}/2\mathbb{Z}$-simple fibration.

We can now restrict our attention only to the sheaf $R^0\sigma_{0*}\mathbb{Z}/2\mathbb{Z}$. For every point $p \in B_0$, $\sigma^{-1}(p)$ consists of the 8 points in the image of $\frac{1}{2}\Lambda_p$ inside $T^*_pB_0/\Lambda_p$. Notice that $\sigma^{-1}(p)$ has a group structure isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Denote this group by $\mathcal{G}_p$ and let $\mathcal{G}$ be the sheaf over B_0 whose stalk is $\mathcal{G}_p$. Observe that

$$\mathcal{G} \cong R^2 f_{0*}\mathbb{Z}/2\mathbb{Z}, \tag{7}$$

in fact $\mathcal{G}$ is naturally isomorphic to $(R^1 f_{0*}\mathbb{Z}/2\mathbb{Z})^\vee$.

Now let us denote by $\mathcal{G}'$ the sheaf $R^0\sigma_{0*}\mathbb{Z}/2\mathbb{Z}$ and observe that $\mathcal{G}'_p$ is just the set of maps from $\mathcal{G}_p$ to $\mathbb{Z}/2\mathbb{Z}$. Clearly, constant maps are monodromy invariant, but since monodromy acts linearly on $\mathcal{G}_p$, also the map which is 1 at $0 \in \mathcal{G}_p$ and zero elsewhere is monodromy invariant. Let us denote by $\mathcal{C}$ the sheaf generated by the constant maps and this latter map. Since these maps are monodromy invariant, $\mathcal{C}$ is just the constant sheaf $(\mathbb{Z}/2\mathbb{Z})^2$. Also note that $\mathcal{G}^\vee$ is naturally a subsheaf of $\mathcal{G}'$. It can be shown that there is a short exact sequence of sheaves

$$0 \to \mathcal{G}^\vee \oplus \mathcal{C} \to \mathcal{G}' \to \mathcal{G} \to 0.$$

The map $\mathcal{G}' \to \mathcal{G}$ in the above sequence is defined as follows. Let $g \in \mathcal{G}_p$ and denote by $\delta_g \in \mathcal{G}'_p$ the map which is 1 at g and zero elsewhere. One can show that every class in the quotient of $\mathcal{G}'_p$ by $\mathcal{G}^\vee_p \oplus \mathcal{C}_p$ is represented by a δ_g for a unique g. So the map from $\mathcal{G}'_p$ to $\mathcal{G}_p$ maps every element in the class of δ_g to g. It can be shown that this map is linear and that it is a morphism of sheaves. Using (7) and $\mathbb{Z}/2\mathbb{Z}$-simplicity the above exact sequence becomes

$$0 \to (R^1 f_*\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to R^0\sigma_*\mathbb{Z}/2\mathbb{Z} \to R^2 f_*\mathbb{Z}/2\mathbb{Z} \to 0.$$

The fact that the sequence remains short exact after applying i_* follows by directly computing the above maps on elements which are locally monodromy invariant near points $p \in \Delta$. With some abuse of notation, we continue to denote this sequence by

$$0 \to \mathcal{G}^\vee \oplus \mathcal{C} \to \mathcal{G}' \to \mathcal{G} \to 0.$$

Passing to the long exact sequence in sheaf cohomology, we obtain

THEOREM 1. *The sheaves $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}^\vee$ over B satisfy the following long exact sequence:*

$$\begin{array}{rlll} 0 \to H^0(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^0(B, \mathcal{G}') \to & H^0(B, \mathcal{G}) \to & \\ H^1(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^1(B, \mathcal{G}') \to & H^1(B, \mathcal{G}) \xrightarrow{\beta} & \\ H^2(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^2(B, \mathcal{G}') \to & H^2(B, \mathcal{G}) \to 0 & \end{array} \tag{8}$$

Observe that from the Leray spectral sequence for σ, we have that

$$H^j(B, \mathcal{G}') \cong H^j(\Sigma, \mathbb{Z}/2\mathbb{Z}).$$

and from the definitions of $\mathcal{G}$, $\mathcal{G}^\vee$ and $\mathcal{C}$, for $j = 1,2$, we have

$$H^j(B, \mathcal{G}^\vee \oplus \mathcal{C}) \cong H^j(B, \mathcal{G}^\vee) \cong H^j(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}),$$

$$H^j(B, \mathcal{G}) \cong H^j(B, R^2 f_* \mathbb{Z}/2\mathbb{Z}).$$

So we obtain

COROLLARY 1. *Σ has two connected components.*

Proof. Since $H^0(B, \mathcal{G}) = H^0(B, \mathcal{G}^\vee) = 0$ and $\mathcal{C} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the first row splits off from the rest and it tells us that $H^0(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence Σ has two connected components. □

One of the two components is the zero section, therefore diffeomorphic to S^3. Notice also that the second row tells us that $H^1(B, R^1 f_* \mathbb{Z}/2\mathbb{Z})$ injects into $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

COROLLARY 2. *With the above hypotheses*

$$\dim H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \geq \dim H^2(X, \mathbb{Z}/2\mathbb{Z})$$

Moreover if $\beta = 0$, for $j = 1,2$ we would have

$$H^j(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong H^j(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}) \oplus H^j(B, R^2 f_* \mathbb{Z}/2\mathbb{Z})$$

Observe that if $\beta = 0$ then Σ satisfies the equality in the inequality (5) and is therefore maximal. Observe also that in the 2-dimensional case of Example 9, a similar but smaller spectral sequence gives us:

$$H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong H^1(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}) \oplus H^2(B, R^2 f_* \mathbb{Z}/2\mathbb{Z}).$$

Since the total space is a K3 surface and Σ is oriented, the above equality holds since $b_1(\Sigma) = 2h^{1,1}(X) = 20$, which is what we already noticed.

4. An example: Schoen's Calabi–Yau

4.1. The manifold and the fibration

At the time of writing this note, we were able to compute the cohomology of only one example of fixed point locus of an involution of the type described. It comes from a Lagrangian fibration of the so-called Schoen's Calabi–Yau. This manifold was studied in [15], and then described in terms of its associated affine manifold with singularities by Gross in [10]. Kovalev [13] described a 3-torus fibration, which inspired the construction we provide here. Consider $f_1 : Y_1 \to \mathbb{P}^1$ and $f_2 : Y_2 \to \mathbb{P}^1$ two rational elliptic surfaces with a section, such that there does not exist $x \in \mathbb{P}^1$ for which $f_1^{-1}(x)$ and $f_2^{-1}(x)$ are both singular. Then Schoen's Calabi–Yau is the fibred product $Y = Y_1 \times_{\mathbb{P}^1} Y_2$. It satisfies $\chi(X) = 0$, $h^{1,1}(X) = h^{1,2}(X) = 19$. It can also be written as a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ of hypersurfaces of tridegree $(1,3,0)$ and $(1,0,3)$.

A topological construction can be given as follows. Consider a 4-dimensional manifold with boundary $\overline{M}$ which fibres over the closed 2-disc D so that the general fibre is a 2-torus and such that there are 12 singular fibres of Kodaira type I_1 (pinched tori) over interior points of D. Assume also that the boundary of $\overline{M}$ is a trivial 2-torus bundle over $\partial D = S^1$, i.e. $\partial\overline{M} \cong T \times S^1$, where T is a 2-torus. To construct $\overline{M}$ we can proceed as follows. Take an elliptically or Lagrangian fibred $K3$, with 24 singular fibres of Kodaira type I_1. Then consider a simple closed curve γ on the base bounding a 2-disc D containing images of 12 singular fibres, and such that it does not pass through critical points. If, furthermore, we choose γ so that along it the H^1-monodromy of the fibres is trivial, then we can take $\overline{M}$ to be the union of the fibres over D. Another construction can be found in [14]. Now consider the 6-manifold with boundary $\overline{X} = \overline{M} \times T'$, where T' is a 2-torus. Clearly $\overline{X}$ fibres over $D \times S^1$ by taking the product of the given fibration of $\overline{M}$ with the standard S^1 fibration of T. The boundary of $\overline{X}$ is $S^1 \times T \times T'$, where $S^1 \times T$ is the boundary of $\overline{M}$. Consider coordinates on $\partial\overline{X}$ given by $(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2)$, where ϕ_1 is the (angle) coordinate on S^1, (ϕ_2, ϕ_3) and (θ_1, θ_2) are (angle) coordinates on T and T' respectively. Assume that the fibration restricted to $\partial\overline{X}$ is the projection onto the coordinates $(\phi_1, \theta_1) \in \partial D \times S^1$.

Now consider the homeomorphism $\Phi : \partial\overline{X} \to \partial\overline{X}$ given by

$$\Phi(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2) = (\theta_1, \theta_2, -\phi_3, \phi_1, \phi_2).$$

We form the manifold Y by gluing two copies of $\overline{X}$ along their boundary using the homeomorphism Φ, i.e.

$$Y = \overline{X} \sqcup_\Phi \overline{X}.$$

It turns out that Y is diffeomorphic to Schoen's Calabi–Yau (see Gross [10] and Kovalev [13]). Notice that if we glue two copies of the base $D \times S^1$ of the fibration on $\overline{X}$ via the map $(\phi_1, \theta_1) \mapsto (\theta_1, \phi_1)$, then we obtain a 3-sphere S^3 and a 3-torus fibration of Y on S^3 induced from the fibration on $\overline{X}$. One can show, using the results of [2], that this fibration can be turned into a (smooth) Lagrangian fibration, by compactifying the affine manifold with singularities constructed by Gross [10, §4].

4.2. The involution and the fixed point locus

We now describe the involution on Y. On $\overline{M}$ we can construct a fibrepreserving involution, simply by considering the involution on the K3 as described in Example 9 and restricting it to $\overline{M}$. On T' we take the involution which preserves the fibres of the fibration on S^1, i.e. in coordinates (θ_1, θ_2), the involution is $(\theta_1, \theta_2) \mapsto (\theta_1, -\theta_2)$. Then on $\overline{X} = \overline{M} \times T'$ we take the product involution. It clearly descends to an involution on Y.

Let us now describe the fixed point locus of this involution. The fixed point locus of the involution on $\overline{M}$ is the disjoint union of a 2-disc, which we denote by S_0 (corresponding to the zero section on the $K3$) and a genus 4 surface S_1 with 3 open discs removed. The boundary of the 2-disc and the three copies of S^1 forming the boundary of S_1 are mapped by the fibration to the boundary of the base ∂D. In fact, we may assume that, with respect to the coordinates (ϕ_1, ϕ_2, ϕ_3) of $\partial \bar{M} = S^1 \times T$ the involution is $(\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, -\phi_3)$ so the four circles are given by $(\phi_1, 0, 0)$ (which is the boundary of S_0), $(\phi_1, 1/2, 0)$, $(\phi_1, 0, 1/2)$, $(\phi_1, 1/2, 1/2)$. The latter three form the boundary of S_1. Now on T', the fixed locus of the involution is given by a pair of circles, corresponding to $(\theta_1, 0)$ and $(\theta_1, 1/2)$. Therefore the fixed point locus of the involution on $\bar{M} \times T'$ is given by two copies of $S_0 \times S^1$ and two copies of $S_1 \times S^1$, i.e.

$$\bar{\Sigma} = (S_0 \times S^1) \sqcup (S_0 \times S^1) \sqcup (S_1 \times S^1) \sqcup (S_1 \times S^1).$$

Then, the fixed point locus of the involution on Y is obtained by gluing together two copies of $\bar{\Sigma}$ via the the homeomorphism Φ restricted to the boundary components of $\bar{\Sigma}$. The result is a 3-manifold Σ with two connected components, one of which is a 3-sphere. According to our computations (based on the Mayer–Vietoris Theorem) $H^1(\Sigma, \mathbb{Z})$ is $\mathbb{Z}^{34}$. Since $H^2(Y, \mathbb{Z}) = \mathbb{Z}^{19}$ and $H^3(Y, \mathbb{Z}) = \mathbb{Z}^{40}$, in the long exact sequence of Theorem 1 applied to Y and Σ, we have

$$H^1(B, \mathcal{G}) = (\mathbb{Z}/2\mathbb{Z})^{19},$$

and

$$\dim \ker \beta = 15.$$

So this is an example where β is not zero.

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Workshop on Hodge Theory and Algebraic Geometry

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THE SECOND GAUSSIAN MAP FOR CURVES: A SURVEY[*]

Abstract. We give an overview of results on the second Gaussian map for curves in relation to the second fundamental form of the period map. We will also concentrate on the case in which the curve is contained in an abelian surface, or in a K3 surface. This is an expanded version of the talk given at the "Workshop on Hodge Theory and Algebraic Geometry" in Povo (Trento) on 4–5 September 2009.

1. Introduction

Let A_g be the moduli space of principally polarized abelian varieties of dimension g and let $j : M_g \to A_g$ be the period map sending a curve to its Jacobian. It is an interesting and classical problem to understand the geometry of the image of M_g in A_g.

On A_g there is a natural metric coming from the unique (up to scalar) $Sp(2g,\mathbb{R})$ invariant metric on the Siegel space $H_g \simeq Sp(2g,\mathbb{R})/U(g)$ of which A_g is the quotient by $Sp(2g,\mathbb{Z})$. In [12] we studied the metric on M_g induced by this metric via the period map, which we call the Siegel metric. In [14] an explicit expression for the second fundamental form of the immersion j is given and it is proven that the second fundamental form lifts the second Gaussian map $\gamma_C^2 : I_2(K_C) \to H^0(C, 4K_C)$, as stated in an unpublished paper of Green and Griffiths (cf. [16]).

This paper is a survey of results that were obtained in collaboration with Elisabetta Colombo and Giuseppe Pareschi in [10, 12, 11, 13] on the second Gaussian map γ_C^2 of a curve C and its relation to the second fundamental form of the period map.

More precisely, in Section 2 we recall the definition of the Gaussian (or Wahl) maps, while in Section 3 we describe some results of [12] concerning the computation of the holomorphic sectional curvature of M_g, endowed with the Siegel metric, along the tangent directions given by the Schiffer variations, in terms of the second Gaussian map γ_C^2.

In Section 4 we first explain some results of [10], namely the computation of the rank of γ_C^2 on the hyperelliptic and trigonal loci. Then from these results and from the fact (proven in [10]) that for a non-hyperelliptic and non-trigonal curve of genus $g \geq 5$, the image of γ_C^2 is base point free, we derive some properties of the holomorphic sectional curvature of M_g. In particular along a Schiffer variation ξ_P the holomorphic sectional curvature $H(\xi_P)$ of M_g is strictly smaller than the holomorphic sectional curvature of A_g unless P is either a Weierstrass point of a hyperelliptic curve or a ramification point of the g_3^1 on a trigonal curve. In these last cases, $H(\xi_P) = -1$.

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In Section 5 we restrict our attention to the hyperelliptic locus and we show that the holomorphic sectional curvature of the hyperelliptic locus, computed along the Schiffer variations at the Weierstrass points of the curve is identically equal to -1.

In Section 6 we describe some results of [11] on the second Gaussian map of a curve in a K3 surface. More precisely we explain the strategy of the proof of the main result of [11], which asserts that for a general hyperplane section of a general polarized K3 surface of genus $g > 280$, γ^2_C is surjective. From this and with the help of some examples that we found in [10], we deduce that for the general curve of genus $g > 152$, γ^2_C is surjective.

Very recently, Calabri, Ciliberto and Miranda in [5] have shown that γ^2_C has maximal rank for the general curve of any genus g, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$. Their proof is achieved by degeneration to a stable binary curve, i.e. a the union of two rational curves meeting at $g+1$ points.

Finally in Section 7 we report on results on the first and the second Gaussian map for a curve on an abelian surface obtained in [13].

The main result of [13] roughly says that if a curve C is contained in an abelian surface, the corank of γ^2_C is at least 2, hence it is never surjective.

Recall that by an important theorem of Wahl ([27]), we know that if a curve is a hyperplane section of a K3 surface, the first Gaussian (or Wahl) map is not surjective (cf. also [3]).

In [13] we also proved that for a “sufficiently ample” curve C contained in an abelian surface the first Wahl map is surjective (see Theorem 7.1 for a precise statement).

Finally recall Ciliberto-Harris-Miranda’s theorem ([6], see also Voisin’s proof in [24]), stating that the first Wahl map of the generic curve of genus g is surjective as soon as this is numerically possible, i.e. for $g \geq 10$, with the exception of $g = 11$. For $g < 10$ and $g = 11$ it is known that the generic curve lies on a K3 surface ([20]).

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2. Definition of Gaussian maps

Let Y be a smooth complex projective variety and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let L and M be line bundles on Y. For a non-negative integer k, the *k-th Gaussian map* associated to these data is the restriction to diagonal map

$$(1)\quad \gamma^k_{L,M} : H^0(Y\times Y, I^k_{\Delta_Y}\otimes L\boxtimes M) \to H^0(Y, I^k_{\Delta_Y|\Delta_Y}\otimes L\otimes M) \cong H^0(Y, S^k\Omega^1_Y\otimes L\otimes M).$$

Usually *first* Gaussian maps are simply referred to as *Gaussian maps*. The exact sequence

$$(2)\qquad 0 \to I^{k+1}_{\Delta_Y} \to I^k_{\Delta_Y} \to S^k\Omega^1_Y \to 0$$

(where $S^k\Omega^1_Y$ is identified to its image via the diagonal map), twisted by $L \boxtimes M$, shows that the domain of the k-th Gaussian map is the kernel of the previous one:

$$\gamma^k_{L,M} : \ker \gamma^{k-1}_{L,M} \to H^0(S^k\Omega^1_Y \otimes L \otimes M).$$

In what follows, we will deal with Gaussian maps of order one and two, assuming also that the two line bundles L and M coincide. The map γ^0_L is the multiplication map of global sections

$$H^0(X,L) \otimes H^0(X,L) \to H^0(X,L^2) \tag{3}$$

which obviously vanishes identically on $\wedge^2 H^0(L)$. Consequently,

$$H^0(Y \times Y, I_{\Delta_Y} \otimes L \boxtimes L)$$

decomposes as $\wedge^2 H^0(L) \oplus I_2(L)$, where $I_2(L)$ is the kernel of $S^2H^0(X,L) \to H^0(X,L^2)$. Since γ^1_L vanishes on symmetric tensors, one writes

$$\gamma^1_L : \wedge^2 H^0(L) \to H^0(\Omega^1_Y \otimes L^2). \tag{4}$$

Again, $H^0(Y \times Y, I^2_{\Delta_Y} \otimes L \boxtimes L)$ decomposes as the sum of $I_2(L)$ and the kernel of (4). Since γ^2_L vanishes identically on skew-symmetric tensors, one usually writes

$$\gamma^2_L : I_2(L) \to H^0(S^2\Omega^1_Y \otimes L^2). \tag{5}$$

(In general, Gaussian maps of even, respectively odd, order vanish identically on skew-symmetric, respectively symmetric, tensors.) The second Gaussian map γ^2_C is the second Gaussian map of the canonical line bundle. In what follows we will also deal with the first Wahl map, γ^1_C, which is the first Gaussian map of the canonical bundle. We have:

$$\gamma^1_C : \wedge^2 H^0(K_C) \to H^0(3K_C)$$

$$\gamma^2_C : I_2(K_C) \to H^0(4K_C).$$

Let us now also give the expressions of γ^1_C and γ^2_C in local coordinates. Fix a basis $\{\omega_i\}$ of $H^0(K_C)$. In local coordinates, assume that $\omega_i = f_i(z)dz$. Then we have

$$\gamma^1_C(\omega_i \wedge \omega_j) = (f'_i f_j - f'_j f_i)(dz)^3.$$

Let $Q \in I_2(K_C)$, $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$, recall that $\sum_{i,j} a_{ij} f_i f_j \equiv 0$, and since $a_{i,j}$ are symmetric, we also have $\sum_{i,j} a_{ij} f'_i f_j \equiv 0$. The local expression of $\gamma^2_C(Q)$ is

$$\gamma^2_C(Q) = \sum_{i,j} a_{ij} f''_i f_j (dz)^4 = -\sum_{i,j} a_{ij} f'_i f'_j (dz)^4. \tag{6}$$

3. The Siegel metric

Let M_g, respectively $M_g^{(n)}$, be the moduli space of smooth genus g curves, respectively of smooth genus g curves with a fixed n-level structure.

Let A_g, respectively $A_g^{(n)}$, be the moduli space of g-dimensional principally polarized Abelian varieties, respectively of g-dimensional principally polarized Abelian varieties with a n-level structure.

Denote by

$$H_g := \{Z \in M(g,\mathbb{C}) \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$$

the Siegel space so that A_g is the quotient of H_g by the action of $Sp(2g,\mathbb{Z})$ and $A_g^{(n)}$ is the quotient of H_g by $\ker(Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/n\mathbb{Z}))$. Denote by $j : M_g \to A_g$, and $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ the period maps which send a curve to its Jacobian.

The Torelli theorem states that j is injective, while $j^{(n)}$ is two-to-one on the image and ramified over the hyperelliptic locus. In fact multiplication by -1 in $H^1(C,\mathbb{Z}) = H^1(JC,\mathbb{Z})$, where JC is the Jacobian of the curve C, is induced by an automorphism of abelian varieties but not by an automorphism of non-hyperelliptic curves. The local Torelli theorem says that outside the hyperelliptic locus and restricted to the hyperelliptic locus the period map is an immersion (cf. [21]). From now on we shall work on $M_g^{(n)}$ and $A_g^{(n)}$, with $n \geq 3$, since they are smooth, everything works in the same way on M_g and A_g but in the orbifold context.

We will now define the Siegel metric.

The Siegel space H_g is a homogeneous space and it can be seen as the quotient $Sp(2g,\mathbb{R})/U(g)$. We call the unique (up to scalar) invariant metric Siegel metric.

Let F be the homogeneous vector bundle on H_g associated to the standard g-dimensional representation of $U(g,\mathbb{C})$. The Hodge metric h on F is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle F. Moreover through the identification

$$\Omega^1_{H_g} \simeq S^2F$$

the Hodge metric on F defines the Siegel metric on H_g .

The Siegel metric on H_g defines a metric on $A_g^{(n)}$ and A_g and, through the period map, an induced metric on $M_g^{(n)}$ and M_g outside the hyperelliptic locus, and on the hyperelliptic locus itself. We call all these metrics Siegel metrics.

These metrics can be described in terms of polarized variation of Hodge structures. More precisely, on $A_g^{(n)}$ we have the universal family $\phi : \mathcal{A} \to A_g^{(n)}$, and the polarized variation of Hodge structures associated to the local system $R^1\phi_*\mathbb{Z}$. The associated Hodge bundle $\mathcal{F}^1$ can be identified with $\phi_*(\Omega^1_{\mathcal{A}|A_g^{(n)}})$, where $\Omega^1_{\mathcal{A}|A_g^{(n)}}$ is the sheaf of relative holomorphic one forms. The polarization induces a Hermitian metric on $R^1\phi_*\mathbb{C}$ and on $\mathcal{F}^1$, which we call the Hodge metric. In fact the pullback of $\mathcal{F}^1$ on H_g is the bundle F and the pullback of the metric is the Hodge metric on F. Hence the Siegel metric is induced by the Hodge metric through the identification $S^2\mathcal{F}^1 \cong \Omega^1_{A_g^{(n)}}$.

On $M_g^{(n)}$ we have the universal family $\psi : \mathcal{C} \to M_g^{(n)}$ with induced relative dualizing sheaf $K_{\mathcal{C}|M_g^{(n)}}$. The local system $R^1\psi_*\mathbb{Z}$ coincides with the pullback of $R^1\phi_*\mathbb{Z}$ through the period map: at a point $[C] \in M_g^{(n)}$, we have $H^1(C,\mathbb{Z}) \cong H^1(JC,\mathbb{Z})$. The non-degenerate Hermitian product on $H^1(C,\mathbb{C})$, defined by the polarization is the following: for any $[\eta],[\xi] \in H^1(C,\mathbb{C})$, we have

$$\langle[\eta],[\xi]\rangle = i\int_C \eta \wedge \overline{\xi}.$$

The Hodge bundle can be identified with $\psi_*(K_{\mathcal{C}|M_g^{(n)}})$, and the corresponding Hodge metric yields a metric on $S^2\mathcal{F}^1 \cong j^{(n)^*}\Omega^1_{A_g^{(n)}}$, hence on $j^{(n)^*}\mathcal{T}_{A_g^{(n)}}$, and by restriction the Siegel metric on $\mathcal{T}_{M_g^{(n)}}$.

We finally observe that for the sake of simplicity we defined the Siegel metric on the fine moduli space $M_g^{(n)}$, but we also have a Siegel metric on M_g viewed as an orbifold.

Recall that outside the hyperelliptic locus we have the sequence of tangent bundles:

$$0 \to \mathcal{T}_{M_g^{(n)}} \to j^{(n)*}\mathcal{T}_{A_g^{(n)}} \xrightarrow{\pi} \mathcal{N} \to 0, \tag{7}$$

whose dual, under the identifications

$$j^{(n)*}\Omega^1{}_{A_g^{(n)}} \cong S^2(\psi_*K_{\mathcal{C}|M_g^{(n)}}), \qquad \Omega^1{}_{M_g^{(n)}} \cong \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}}),$$

is

$$0 \to I_2 \to S^2(\psi_*K_{\mathcal{C}|M_g^{(n)}}) \xrightarrow{m} \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}}) \to 0, \tag{8}$$

where $I_2 := \mathcal{N}^*$ and m is the multiplication map.

The Hermitian connection of the variation of Hodge structures $\mathcal{R}^1\psi_*\mathbb{C}$, the Gauss-Manin connection, defines a Hermitian connection on $\mathcal{F}^1 = \psi_*K_{\mathcal{C}|M_g^{(n)}}$, thus on $\mathcal{F}^{1^*}$, as well as $S^2\mathcal{F}^1$ and $S^2\mathcal{F}^{1^*} \simeq j^{(n)*}\mathcal{T}_{A_g^{(n)}}$, which we denote by ∇.

The exact sequence (7) defines a second fundamental form,

$$\sigma \in \mathrm{Hom}(\mathcal{T}_{M_g^{(n)}}, \mathcal{N} \otimes \Omega^1_{M_g^{(n)}}),\ \sigma : s \mapsto \pi(\nabla(s)).$$

Similarly the exact sequence (8) defines the second fundamental form

$$\rho \in \mathrm{Hom}\left(I_2, \psi_*(K^2_{\mathcal{C}|M_g^{(n)}}) \otimes \Omega^1_{M_g^{(n)}}\right).$$

Clearly σ yields a linear map $\tilde{\sigma} : \mathcal{T}_{M_g^{(n)}} \otimes \mathcal{T}_{M_g^{(n)}} \to \mathcal{N}$, and $\rho : I_2 \to \Omega^1_{M_g^{(n)}} \otimes \Omega^1_{M_g^{(n)}}$ is the dual of $\tilde{\sigma}$.

In [12] we computed the curvature form R of $\mathcal{T}_{M_g^{(n)}}$ in terms of the curvature form $\tilde{R}$ of $j^{(n)*}(\mathcal{T}_{A_g^{(n)}})$ and the second fundamental form σ. In fact we have

$$\langle R(s),t\rangle = \langle \tilde{R}(s),t\rangle - \langle \sigma(s),\sigma(t)\rangle, \tag{9}$$

where s,t are local sections of $\mathcal{T}_{M_g^{(n)}}$.

The exact sequence (7) at $[C] \in M_g^{(n)}$ is

$$0 \to H^1(T_C) \to S^2(H^0(K_C))^* \to I_2(C)^* \to 0, \tag{10}$$

thus σ yields a homomorphism

$$\sigma : H^1(T_C) \to \mathrm{Hom}(I_2(K_C), H^0(2K_C)). \tag{11}$$

Analogously, at $[C] \in M_g^{(n)}$ the exact sequence (8) is:

$$0 \to I_2(K_C) \to S^2(H^0(K_C)) \xrightarrow{m} H^0(2K_C) \to 0, \tag{12}$$

hence the second fundamental form ρ gives a homomorphism

$$\rho : I_2(K_C) \to \mathrm{Hom}(H^1(T_C), H^0(2K_C)) \tag{13}$$

and for every $v \in H^1(T_C)$, and for every $Q \in I_2(C)$, we have

$$\sigma(v)(Q) = \rho(Q)(v).$$

In [12], to compute the curvature form of $\mathcal{T}_{M_g^{(n)}}$ at $[C] \in M_g^{(n)}$, we used some results of [14]. Namely in [14] it is proven that the map ρ, whose image is in $S^2H^0(2K_C)$, is a lifting of the second Gaussian map γ_C^2, as in the following diagram

(14)
$$\begin{tikzcd} I_2(K_C) \arrow[r, "\rho"] \arrow[dr, "\gamma_C^2"] & S^2H^0(2K_C) \arrow[d, "m"] \\ & H^0(4K_C) \end{tikzcd}$$

where the map m is given by multiplication.

Moreover in [14] an explicit computation of the image of ρ at the tangent direction given by a Schiffer variation at a point P of C is given in terms of the second Gaussian map, namely we have:

$$\xi_P(\rho(Q)(\xi_P)) = 2\pi i \gamma_C^2(Q)(P), \tag{15}$$

for every quadric $Q \in I_2$.

Let us briefly recall the definition of ξ_P. Consider the exact sequence

$$0 \to T_C \to T_C(P) \to T_C(P)_{|P} \to 0.$$

Notice that $H^0(T_C(P)_{|P}) \cong \mathbb{C}$. If we denote the coboundary map by

$$\delta : H^0(T_C(P)_{|P}) \to H^1(T_C),$$

we have $\dim(\mathrm{Im}(\delta)) = 1$. Any non-zero element ξ_P in $\mathrm{Im}(\delta)$ is called a Schiffer variation. Let us a choose a local coordinate z in a neighborhood of P. Under the Dolbeault isomorphism $H^1(T_C) \cong H^{0,1}(T_C)$, it is represented by the form $\theta_P = \frac{1}{z}\overline{\partial} b_P \otimes \frac{\partial}{\partial z}$, where b_P is a bump function around P. Notice that if we choose b_P to be one in a neighborhood of P for this choice of local coordinate z, ξ_P depends only on the choice of z and in fact also formula (15) depends on z.

Using these results, in [12] we obtained a closed expression for the holomorphic sectional curvature of $\mathcal{T}_{M_g^{(n)}}$ at $[C] \in M_g^{(n)}$ along the tangent directions given by the Schiffer variations ξ_P. More precisely, the following holds.

THEOREM 1. *([12]) The holomorphic sectional curvature of* $\mathcal{T}_{M_g^{(n)}}$ *at* $[C] \in M_g^{(n)}$ *computed at the tangent vector* ξ_P *is given by*

$$\begin{aligned} H(\xi_P) &= \frac{1}{\langle \xi_P, \xi_P \rangle \langle \xi_P, \xi_P \rangle} \langle R(\xi_P), \xi_P \rangle (\xi_P, \overline{\xi_P}) \\ &= -1 - \frac{1}{64\pi^2 (\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2, \end{aligned}$$

where $\{Q_i\}$ *is an orthonormal basis of* I_2, $\{\omega_j\}$ *is an orthonormal basis of* $H^0(K_C)$ *and in a local coordinate z around P, we have* $\omega_j = f_j(z)dz$.

In the above formula, -1 is the value of the holomorphic sectional curvature of $A_g^{(n)}$ calculated along the tangent directions at $[C] \in M_g^{(n)}$ given by ξ_P, for all $P \in C$, while the term $-\frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2$ represents the contribution given by the second fundamental form.

4. Second Gaussian map and curvature

We first recall some results on the second Gaussian map obtained in [10].

Assume that C is either a hyperelliptic curve of genus $g \geq 3$, or a trigonal curve of genus $g \geq 4$. Let $|F|$ denote the g_2^1 in the hyperelliptic case, the g_3^1 in the trigonal case. Let $\phi_F : C \to \mathbb{P}^1$ be the induced morphism and $\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ be the Veronese embedding, so that in the hyperelliptic case $\phi_K = \nu \circ \phi_F$, where ϕ_K is the canonical map. Observe that in the hyperelliptic case the hyperelliptic involution τ acts as $-Id$ on $H^0(K_C)$, so we have an exact sequence

$$0 \to I_2(K_C) \to S^2(H^0(K_C)) \to H^0(2K_C)^+ \to 0,$$

where $H^0(2K_C)^+$ denotes the τ-invariant part of $H^0(2K_C)$ whose dimension is $(2g-1)$ and $I_2(K_C)$ is the vector space of the quadrics containing the rational normal curve.

Set $L := K_C - F$, and fix a basis $\{x,y\}$ of $H^0(F)$, and a basis $\{t_1,\dots,t_r\}$ of $H^0(L)$ both in the hyperelliptic and in the trigonal case. We have a linear map

$$\psi : \Lambda^2(H^0(L)) \to I_2, \quad t_i \wedge t_j \mapsto Q_{ij} = xt_i \odot yt_j - xt_j \odot yt_i.$$

In both cases the linear map $\psi : \Lambda^2(H^0(L)) \to I_2$ is an isomorphism as can be easily checked or found in [1].

An easy computation in local coordinates shows that we have

$$\gamma^2_C(Q_{ij}) = \gamma^1_F(x \wedge y)\gamma^1_L(t_i \wedge t_j),$$

hence, for any quadric Q of rank four, we have

$$\gamma^2_C(Q) = \gamma^1_F(x \wedge y)\gamma^1_L(\psi^{-1}(Q)).$$

So, if we denote by $q_1,\dots,q_l$ the ramification points of either the g^1_2, or of the g^1_3, we see that the image of γ^2_C is contained in $H^0(4K_C - (q_1 + \cdots + q_l))$ and $\operatorname{rank}(\gamma^2_C) = \operatorname{rank}(\gamma^1_L)$. In fact,

$$\operatorname{div}(\gamma^2_C(Q)) = \operatorname{div}(\gamma^1_F(x \wedge y)) + \operatorname{div}(\gamma^1_L(\psi^{-1}(Q))) = q_1 + \cdots + q_l + \operatorname{div}(\gamma^1_L(\psi^{-1}(Q)).$$

Therefore $\gamma^2_C(Q)(q_i) = 0$ for all $i = 1,\dots,l$.

Using the above observations and the computation of the rank of the first Gaussian map of the canonical linear series for hyperelliptic curves done in [9], in [10] we proved the following

PROPOSITION 1. *([10, Lem. 4.1, Prop. 4.2]) Let C be a hyperelliptic curve of genus $g \geq 3$. Then the rank of γ^2_C is $2g-5$ and its image is contained in*

$$H^0(4K_C - (q_1 + \cdots + q_{2g+2})),$$

where $\{q_1,\dots,q_{2g+2}\}$ are the Weierstrass points.

Assume now that C is non-hyperelliptic trigonal curve of genus $g \geq 4$. Let $|F|$ be the g^1_3 on C, assume $F = p_1 + p_2 + p_3$, $p_i \in C$. Denote by $L = K_C - F = K_C - p_1 - p_2 - p_3$, $\deg(L) = 2g-5$, $h^0(L) = g-2$. Thus $H^0(L) \subset H^0(K_C)$ and $\gamma^1_L = \gamma^1_{K|\Lambda^2 H^0(K_C - p_1 - p_2 - p_3)}$. In [9] it is proven that for the general trigonal curve of genus $g \geq 4$, $\dim(\operatorname{coker}(\gamma^1_K)) = g+5$, moreover specific examples of trigonal curves (whose genera are all equal to 1 modulo 3) such that the corank of γ^1_K is $g+5$ are exhibited. Using results of [15], in [4] Brawner proved that $\dim(\operatorname{coker}(\gamma^1_K)) = g+5$ for any trigonal curve of genus $g \geq 4$.

In [10] we determined the rank of γ^2_C for trigonal curves, computing the rank of γ^1_L and generalizing the computation done in [15] and [4] for γ^1_K.

THEOREM 2. *([10]) For any trigonal non-hyperelliptic curve C of genus $g \geq 4$, the image of γ^2_C is contained in $H^0(4K_C - (q_1 + \cdots + q_{2g+4}))$, where $q_1 + \cdots + q_{2g+4}$ is the ramification divisor of the g^1_3.*

If $g \geq 8$, the rank of γ^2_C is $4g-18$.

We also recall

THEOREM 3. *([10]) Assume that C is smooth curve of genus $g \geq 5$, which is non-hyperelliptic and non-trigonal. Then for any $P \in C$ there exists a quadric $Q \in I_2$ such that $\gamma_C^2(P) \neq 0$. Equivalently, for all $P \in C$,* $\mathrm{Im}(\gamma_C^2) \not\subset H^0(4K_C - P)$.

Assume $[C] \in M_g^{(n)}$, with $g \geq 4$, C non-hyperelliptic. Then Theorem 1 allows us to define a function $F : C \to \mathbb{R}$, given by the holomorphic sectional curvature evaluated along the tangent vectors given by the Schiffer variations:

$$F(P) = H(\xi_P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2 \leq -1,$$

where $\{Q_i\}$ is an orthonormal basis of $I_2(K_C)$, $\{\omega_j\}$ is an orthonormal basis of $H^0(K_C)$, and $\omega_j = f_j(z)dz$ is a local expression around P.

PROPOSITION 2. *([12])*

- *If $g = 4$, the set of points $P \in C$ such that $F(P) = -1$ is finite, which implies that F is non-constant.*
- *If $g \geq 5$ and C is neither hyperelliptic nor trigonal, then $F(P) < -1$ for all $P \in C$.*
- *If C is a trigonal curve of genus ≥ 4, $F(P) = H(\xi_P) = -1$ for every $P \in C$ which is a ramification point of the g_3^1, while there exist points $x \in C$ such that $F(x) < -1$, hence F is not constant.*

Proof. Assume C has genus 4, then the dimension of I_2 is one and I_2 can be generated by a quadric Q of rank 4 which has norm 1. So $F(P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} |\gamma_C^2(Q)(P)|^2$ for all $P \in C$, hence there is a finite number of points P such that $\gamma_C^2(Q)(P) = 0$, so in these points we have $F(P) = -1$, while $F(P) < -1$ elsewhere .

As regards the second statement, we observe that $F(P) = -1$ if and only if $\gamma_C^2(Q_i)(P) = 0$ for all i, where $\{Q_i\}$ is an orthonormal basis of I_2. But then we must have $\gamma_C^2(Q)(P) = 0$ for all $Q \in I_2$. So the proof follows by Theorem 3.

The last statement follows from Theorem 2 and from the observation that if x is a point in C such that $\gamma_C^2(Q_1)(x) \neq 0$, we have $F(x) < -1$. ☐

REMARK 1. The previous statements imply that for any curve $C \in M_g^{(n)}$, not hyperelliptic, nor trigonal, for every point $P \in C$ the holomorphic sectional curvature of $M_g^{(n)}$, at C along the tangent directions given by ξ_P is strictly smaller than the holomorphic sectional curvature of $A_g^{(n)}$.

On the other hand, in the trigonal case, along the Schiffer variations at the ramification points of the g_3^1, (which are a basis of the tangent space to the trigonal locus) the holomorphic sectional curvature of $M_g^{(n)}$, coincides with the holomorphic sectional curvature of $A_g^{(n)}$.

5. The hyperelliptic locus

We will now explain some results obtained in [10] and [12] on the hyperelliptic locus $HE_g \subset M_g^{(n)}$. Recall that by local Torelli, the restriction of the period map to HE_g is an injective immersion (cf. [21]). Therefore we have the exact sequence

$$0 \to \mathcal{T}_{HE_g} \to \mathcal{T}_{A_g^{(n)}|HE_g} \to \mathcal{N}_{HE_g|A_g^{(n)}} \to 0,$$

and we denote by

$$\sigma_{HE} : \mathcal{T}_{HE_g} \to \mathrm{Hom}(\mathcal{T}_{HE_g}, \mathcal{N}_{HE_g|A_g^{(n)}})$$

the associated second fundamental form and by ρ_{HE} the second fundamental form of the dual exact sequence. At the point $[C] \in HE_g$ the dual exact sequence is

$$0 \to I_2 \to S^2(H^0(K_C)) \to H^0(2K_C)^+ \to 0,$$

where $H^0(2K_C)^+$ is the invariant part of $H^0(2K_C)$ under the hyperelliptic involution and I_2 is the vector space of the quadrics containing the rational normal curve, so that

$$\rho_{HE} : I_2 \to \mathrm{Hom}(\mathcal{T}_{HE_g,[C]}, H^0(2K_C)^+).$$

We recall that the set of Schiffer variations at the Weierstrass points P_i generates $\mathcal{T}_{HE_g,[C]}$.

In [12], we observed that we have a formula which is similar to (15) at a Weierstrass point $P \in C$:

$$\xi_P(\rho_{HE}(Q)(\xi_P)) = \gamma_C^2(Q)(P)$$

Let us denote by H_{HE} the holomorphic sectional curvature of $\mathcal{T}_{HE_g}$, if $[C] \in HE_g$ and $P \in C$ is a Weiestrass point, we have the same expression for $H_{HE}(\xi_P)$ as in Theorem 1, namely

$$H_{HE}(\xi_P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2 \tag{16}$$

where $\{Q_i\}$ is an orthonormal basis of I_2 and $\{\omega_j =_{loc.} f_j(z)dz\}$ is an orthonormal basis of $H^0(K_C)$.

So, using Proposition 1, we have the following

COROLLARY 1. *([12]) Let $[C] \in HE_g$, then $H_{HE}(\xi_P) = -1$, for any Weierstrass point $P \in C$.*

Proof. The proof immediately follows from (16) and from Proposition 1. ☐

6. Curves on K3 surfaces and results for the general curve

In this section we will explain some results of [11] on the second Gaussian map for curves on K3 surfaces, from which we have also deduced surjectivity of the 2nd Gaussian map for the general curve of sufficiently high genus. Then we will also discuss a theorem of [5] in which, with different methods, they prove that the second Gaussian map for the general curve C of any genus has maximal rank.

First of all let us recall that Wahl ([27]) has given a deformation theoretic interpretation of the first Gaussian map, showing that if a canonical curve can be extended in projective space as a hyperplane section of a surface which is not a cone, then the first Gaussian map is not surjective.

In particular in [27] it is proven that if a curve lies on a K3 surface, the first Gaussian map cannot be surjective (see also [3]).

The obstruction to the surjectivity of the first Gaussian map for a curve in a K3 surface is given by the extension class of the cotangent sequence

$$0 \to K_C^{-1} \to \Omega^1_{X\,|C} \to K_C \to 0, \tag{17}$$

which is a non-trivial element in the kernel of the dual of the first Gaussian map (see [3]).

To study the second Gaussian map γ^2_C for a curve in a K3 surface X it is natural to consider the "symmetric square" of the cotangent extension

$$0 \to \Omega^1_{X\,|C} \otimes K_C^{-1} \to S^2\Omega^1_{X\,|C} \to K_C^2 \to 0. \tag{18}$$

This does not give any obstruction to the surjectivity of γ^2_C for the general curve in a general K3 surface, while it gives an obstruction if C is any curve in an abelian surface, as it is proven in [13]. In fact in the next section we will discuss a result obtained in [13], that asserts that if C is a curve in an abelian surface X, then the corank of γ^2_C is at least two.

The main result that we obtained for curves in K3 surfaces is the following

THEOREM 4. *([11]) If X is a general polarized K3 surface of degree $2g-2$ with $g > 280$ and C is a general hyperplane section of X, then γ^2_C is surjective.*

Let us explain the strategy of the proof of Theorem 4. We have the following commutative diagram

$$\begin{tikzcd}
I_2(\mathcal{O}_X(C)) \arrow[r, "\gamma^2_{\mathcal{O}_X(C)}"] \arrow[dd, "r"] & H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \arrow[dr, "p_1"] & \\
 & & H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \arrow[dl, "p_2"] \\
I_2(K_C) \arrow[r, "\gamma^2_C"'] & H^0(K_C^4) &
\end{tikzcd} \tag{19}$$

where r and p_1 are restriction maps, and p_2 comes from the conormal extension. More precisely, consider the second symmetric square of the cotangent exact sequence (18) tensored by K_C^2:

$$0 \to \Omega^1_{X|C} \otimes K_C \to S^2\Omega^1_{X|C} \otimes K_C^2 \to K_C^4 \to 0, \tag{20}$$

then we have

$$H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \xrightarrow{p_2} H^0(K_C^4) \to H^1(\Omega^1_{X|C} \otimes K_C) \cong H^0(T_{X|C})^*,$$

hence p_2 is surjective by the following lemma.

LEMMA 1. *If X is a general K3 surface and C a general curve of genus at least* 13 *in the very ample linear system $|\mathcal{O}_X(C)|$ then $H^0(T_{X|C}) = 0$.*

Proof. By the exact sequence given by restriction of T_X to C, $H^0(T_{X|C})$ injects in $H^1(T_X(-C))$, which vanishes by lemma (2.3) of [8]. ☐

The theorem follows if we prove that also the maps $\gamma^2_{\mathcal{O}_X(C)}$ and p_1 are surjective. In fact in [11] we exhibited examples of pairs (X,C) where X is a K3 and C is a very ample curve in X of any genus g sufficiently high ($g \geq 281$) for which $\gamma^2_{\mathcal{O}_X(C)}$ and p_1 are surjective.

To do this we followed the strategy used in [7] to study the first Wahl map. More precisely, from the exact sequence

$$\begin{aligned} 0 \to I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) &\to I^2_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) \\ &\to I^2_{\Delta_X}/I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) \to 0 \end{aligned} \tag{21}$$

and taking global sections, we see that $\gamma^2_{\mathcal{O}_X(C)}$ is surjective if $H^1(I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C))) = 0$.

The idea used in [7] is to consider the blow-up Y of $X \times X$ along the diagonal Δ_X and to use Kawamata–Viehweg vanishing theorem ([18, 23]). Let E be the exceptional divisor and denote by $\pi : Y \to X \times X$ the natural morphism and by $f := p \circ \pi$, $g := q \circ \pi$. Then

$$\begin{aligned} H^1(I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C))) &\cong H^1(Y, f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C))(-3E)) \\ &\cong H^1(Y, f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C)) \otimes K_Y(-4E)), \end{aligned}$$

since $K_Y = \mathcal{O}_Y(E)$. So by the Kawamata–Viehweg vanishing theorem, it suffices to prove that $f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C))(-4E)$ is big and nef.

Now notice that if one decomposes $\mathcal{O}_X(C)$ as $\otimes_{i=1}^4 A_i$, where A_i are line bundles on X, then $L = \otimes_{i=1}^4 (f^*(A_i) \otimes g^*(A_i)(-E))$. To obtain that L is big and nef, we asked suitable conditions on the line bundles A_i, and we studied the sublinear system of $|f^*(A_i) \otimes g^*(A_i)(-E)|$ given by $\mathbb{P}(\Lambda^2(H^0(A_i)))$ (cf. lemma 3.3 of [11]).

Consider now the map

$$p_1 : H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \to H^0(S^2\Omega^1_{X|C} \otimes K_C^2).$$

Clearly p_1 is surjective if $H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C)) = 0$.

To prove this vanishing we observed that, given a decomposition of $\mathcal{O}_X(C)$ as $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$, we have

$$H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C)) = H^1(X \times X, I^2_{\Delta_X}/I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D'))),$$

hence its vanishing is implied by that of $H^1(X \times X, I^2_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D')))$ and of $H^2(X \times X, I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D')))$.

So, with the same argument as above, it suffices to show that $f^*(\mathcal{O}_X(D)) \otimes g^*(\mathcal{O}_X(D'))(-4E)$ is big and nef. The strategy is now to choose $\mathcal{O}_X(D) = \otimes_{i=1}^4 A_i$ and $D' = D + B$ with B nef and effective, and take $C \in |2D + B|$.

The above decompositions are shown on concrete examples of K3 surfaces X and of curves C in X, which are explicitely constructed via their Picard lattices (cf. proposition 3.4 of [11]).

Using this result and some examples given in [10], we deduce the following

COROLLARY 2. *([11]) For the general curve of genus greater than* 152*, the second Gaussian map* γ^2_C *is surjective.*

Proof. By Theorem 4 and the semicontinuity of the corank of γ^2_C, for a general curve of genus greater than 280, γ^2_C is surjective. Surjectivity for the general curve of genus $153 \leq g \leq 280$ can be proved exhibiting examples of curves of genus g with surjective second Gaussian map, which are either hyperplane sections of a polarized K3 surface as in the proof of Theorem 4 given in [11], or lying in the product of two curves as in [10, theorem 3.1].

More precisely let C_1, C_2 be two smooth curves of respective genera g_1, g_2, choose divisors D_i on C_i of degree d_i, $i = 1,2$. Set $X = C_1 \times C_2$, let $C \in |p_1{}^*(D_1) \otimes p_2{}^*(D_2)|$ be a smooth curve, where p_i is the projection from $C_1 \times C_2$ on C_i, then $g(C) = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1 d_2$.

In [10] we proved that if either $g_1, g_2 \geq 2$, $d_i \geq 2g_i + 5$, $i = 1,2$, or $g_1 \geq 2$, $g_2 = 1$, $d_1 \geq 2g_1 + 5$, $d_2 \geq 7$, or $g_2 = 0$, $d_2 \geq 7$, $d_2(g_1 - 1) > 2d_1 \geq 4g_1 + 10$, then γ^2_C is surjective.

Then one has to check directly that these values of $g(C)$ cover all the remaining integers between 153 and 280. ☐

Note that, for dimensional reasons, surjectivity can be expected for a general curve of genus at least 18, and in fact recently in [5] Calabri, Ciliberto and Miranda showed that for the general curve of genus $g \geq 18$, the second Gaussian map is surjective. More precisely they proved the following

THEOREM 5. *([5], Theorem 1). The second Gaussian map* $\gamma^2_C : I_2(K_C) \to$

$H^0(C,4K_C)$ for C a general curve of any genus g has maximal rank, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$.

The proof they gave in [5] relies on the study of the limit of the second Gaussian map when the general curve of genus g degenerates to a general stable binary curve, i.e. the union of two rational curves meeting at $g+1$ points. The theorem then follows by upper semicontinuity. For such a stable binary curve C they explicitly write down the ideal $I_2(K_C)$ and they first describe the 2nd Gaussian map for C modulo torsion, then they deal with the torsion part. By direct computations performed with Maple, they verify the injectivity for a general binary curve of genus $g \leq 17$ and the surjectivity for $g = 18$. Finally they complete the argument by induction on g, for $g \geq 19$.

We have observed in the proof of Corollary 2 that examples of curves whose second Gaussian map is surjective were already given in [10] (for curves in the product of two curves) and we recall that other examples were given in [2] (for complete intersections). Notice that using complete intersections it is not possible to deduce surjectivity for the general curve of any sufficiently high genus, due to restrictions on the genus.

On the other hand, Theorem 4 shows that general curves on K3 surfaces of sufficiently high genus behave as general curves in the moduli space, with respect to the second Gaussian map.

Finally, we also observe that with the method used in [11] to prove Theorem 4 it is impossible to reach the optimal lower bound for the genus of the curve C, in fact the conditions that we gave on the line bundles A_i and the decomposition $\mathcal{O}_X(C) = \mathcal{O}_X(2D+B)$ force the genus of C to be high.

Moreover, the vanishing of $H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C))$ itself, already implies that the curve C must be of genus at least 31, as one can check looking at the restriction of $\Omega^1_X \otimes \Omega^1_X(C)$ to C and the induced cohomology exact sequence.

7. Curves on abelian surfaces

In this section we discuss some results obtained in collaboration with E. Colombo and G. Pareschi in [13] on the first and the second Gaussian map for curves on abelian surfaces.

The first result says that the first Gaussian map for a “sufficiently ample” curve in an abelian surface is surjective. In fact we have the following

THEOREM 6 ([13]). *Let C be a smooth irreducible curve contained in an abelian surface X. Assume that the first Gaussian map of the line bundle $\mathcal{O}_X(C)$ on the surface X is surjective and that the multiplication map*

$$\gamma^0_{X,C} : H^0(X, \mathcal{O}_X(C)) \otimes H^0(C, K_C) \to H^0(K_C^2)$$

is surjective (for example, both conditions hold if $\mathcal{O}_X(C)$ is at least a 5-th power of an ample line bundle on X, see [22]). Then the first Gaussian map of C is surjective.

REMARK 2. Recently in [19] it is shown that for a curve C of genus $g > 145$ sitting on a very general abelian surface X, the first Gaussian map of the line bundle $\mathcal{O}_X(C)$ on the surface X is surjective. Hence by Theorem 6 we obtain the surjectivity of the first Gaussian map of C, and therefore a new proof of the surjectivity of the map γ^1_C for the general curve of genus > 145.

The main theorem of [13] asserts that if a curve C is contained in an abelian surface, the second Gaussian map γ^2_C is not surjective. More precisely, let us introduce the following notation. Given a subspace $W \subset H^0(K_C)$, we will denote

$$S^2W \cdot H^0(K_C^2)$$

the image of $S^2W \otimes H^0(K_C^2)$ in $H^0(K_C^4)$ via the natural multiplication map. If W is 2-dimensional and base point free, the base-point-free pencil trick implies that $S^2W \cdot H^0(K_C^2)$ has codimension 2 in $H^0(K_C^4)$. If C is embedded in abelian surface, then $H^0(\Omega^1_X)$ is naturally a (base-point-free) 2-dimensional subspace of $H^0(K_C)$. We have the following

THEOREM 7. *([13]) Let C be a curve contained in abelian surface X. Then the image of the second Gaussian map γ^2_C is contained in $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$ (notation as above). Therefore the corank of γ^2_C is at least 2. Moreover, if the second Gaussian map of the surface X is surjective, then the image of the map γ^2_C concides with $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.*

The above theorem can also be stated as follows. Given a subspace $V \subset H^1(\mathcal{O}_C)$, let $\overline{V} \subset H^0(K_C)$ be its conjugate.

COROLLARY 3. *([13]) Let $C \subset \mathbb{P}^{g-3}$ be a canonically embedded curve of genus g, obtained from the complete canonical embedding $C \subset \mathbb{P}H^1(\mathcal{O}_C) = \mathbb{P}^{g-1}$ by projection from a line $\mathbb{P}V \subset \mathbb{P}H^1(\mathcal{O}_C)$, $\dim V = 2$. If C is a hyperplane section of an abelian surface $X \subset \mathbb{P}^{g-2}$ then*

$$\mathrm{Im}(\gamma^2_C) \subseteq S^2\overline{V} \cdot H^0(K_C^2).$$

The proofs of the above results rely on cohomological methods, which are similar to the ones used in [3].

Let us sketch the proof of Theorem 7.

First of all let us see the Gaussian maps defined as the H^0 of maps of coherent sheaves on the variety Y. This is achieved as follows: let p and q be the two projections of $Y \times Y$. Applying p_* to the exact sequences (2) tensored by a line bundle M one gets the exact sequences

$$0 \to p_*(I^{k+1}_{\Delta_Y} \otimes q^*M) \to p_*(I^k_{\Delta_Y} \otimes q^*M) \xrightarrow{\varphi^k} S^k\Omega^1_Y \otimes M. \tag{22}$$

The Gaussian maps $\gamma^k_{L,M}$ of (1) are obtained by tensoring with L and taking $H^0(L \otimes \varphi^k)$.

Let us spell out how the Gaussian maps γ^1_C look like in this setting. Let R_C be the kernel of the evaluation map of K_C:

$$0 \to R_C \xrightarrow{f} H^0(K_C) \otimes \mathcal{O}_C \to K_C \to 0 \tag{23}$$

(i.e. sequence (22) for $Y = C$, $M = K_C$, $k = 0$). By (22) (same setting) for $k = 1$ we have the natural map

$$R_C \xrightarrow{g} K_C^2. \tag{24}$$

Tensoring with K_C and taking H^0 one obtains the first Wahl map

$$\gamma_C^1 : H^0(R_C \otimes K_C) \to H^0(K_C^3). \tag{25}$$

One has the exact sequence

$$0 \to I_{\Delta_C}^2 \to \mathcal{O}_{C\times C} \to \mathcal{O}_{\Delta_C^2} \to 0 \tag{26}$$

where Δ_C^2 denotes the first infinitesimal neighborhood.

Tensoring sequence (26) with q^*K_C and applying p_*, where p and q are the two projections from $C \times C$, one gets the exact sequence

$$0 \to R_C^2 \xrightarrow{f'} H^0(K_C) \otimes \mathcal{O}_C \xrightarrow{ev} P_C(K_C) \tag{27}$$

where

$$R_C^2 = p_*(q^*(K_C) \otimes I_{\Delta_C}^2), \tag{28}$$

and

$$P_C(K_C) = p_*(q^*(K_C) \otimes \mathcal{O}_{\Delta_C^2})$$

is the bundle of principal parts of K_C.

REMARK 3. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & . \\
 & R_C^2 \arrow[r, "="] \arrow[d] & R_C^2 \arrow[d] & & \\
0 \arrow[r] & R_C \arrow[r] \arrow[d, "g"] & H^0(K_C) \otimes \mathcal{O}_C \arrow[r] \arrow[d, "ev"] & K_C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & K_C^2 \arrow[r] & P_C(K_C) \arrow[r] & K_C \arrow[r] & 0
\end{tikzcd}$$

Notice that the map g of (24) is surjective if and only if the evaluation map ev is surjective. This is in turn equivalent to the immersivity of the canonical map, which holds if and only if C is non-hyperelliptic.

Sequence (22) for $k=2$, $Y=C$ and $M=K_C$ provides the natural map

$$R^2_C \xrightarrow{g'} K^3_C. \tag{29}$$

Tensoring with K_C and taking H^0 one obtains the second Wahl map

$$\gamma^2_C : H^0(R^2_C \otimes K_C) \to H^0(K^4_C), \tag{30}$$

The coboundary map of the exact sequence (20),

$$f_{e'} : H^0(K^4_C) \to H^1(\Omega^1_X \otimes K_C) \cong H^1(K_C)^{\oplus 2} \cong \mathbb{C}^{\oplus 2}$$

is identified, by Serre duality, to the extension class $e' \in \mathrm{Ext}^1(K^3_C, \Omega^1_{X\,|C})$ of sequence (18). The first part of the statement of Theorem 7 is equivalent to:

$$f_{e'} \circ \gamma^2_C = 0. \tag{31}$$

Let us work in the dual setting.

Applying $\mathrm{Ext}^1(\cdot, \Omega^1_{X\,|C})$ to the map g' of (29) one gets the map

$$\phi : \mathrm{Ext}^1(K^3_C, \Omega^1_{X\,|C}) \to \mathrm{Ext}^1(R^2_C, \Omega^1_{X\,|C})$$

(which is identified to two copies of the dual map of the second Wahl map) and it is easily seen that (31) is equivalent to the fact that

$$\phi(e') = 0. \tag{32}$$

Applying $\mathrm{Ext}^1(\cdot, \Omega^1_{X|C})$ to the map f' of (27) we get the map

$$\psi : \mathrm{Hom}(H^0(K_C), H^1(\Omega^1_{X|C})) = \mathrm{Ext}^1(H^0(K_C) \otimes \mathcal{O}_C, \Omega^1_{X|C}) \to \mathrm{Ext}^1(R^2_C, \Omega^1_{X|C}).$$

Now let us denote by $\tilde{\delta}$ the composition of the coboundary map $H^0(K_C) \to H^1(\mathcal{O}_X)$ of the standard exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to K_C \to 0,$$

and the map $H^1(d) : H^1(\mathcal{O}_X) \to H^1(\Omega^1_{X|C})$, induced by the derivation $d : \mathcal{O}_X \to \Omega^1_X$. Note that $H^1(d)$ is the zero map since the Hodge–Frölicher spectral sequence degenerates at level of E_1. Hence the map $\tilde{\delta}$ is equal to zero.

To conclude the proof of the first part of Theorem 7, we showed the following result, whose proof can be found in [13].

CLAIM. $\phi(e') = \psi(\tilde{\delta}) = 0$.

The last part of the statement follows from the commutative diagram (constructed as diagram (19))

(33)
$$\begin{tikzcd}
I_2(\mathcal{O}_X(C)) \arrow[r, "\gamma^2_{\mathcal{O}_X(C)}"] \arrow[dd] & H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \arrow[dr] & \\
& & H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \arrow[dl] \\
I_2(K_C) \arrow[r, "\gamma^2_C"'] & H^0(K_C^4) &
\end{tikzcd}$$

This completes the proof of Theorem 7.

Finally, in [22, Th. 2.2] it is shown that, if $\mathcal{O}_X(C)$ is at least a 7-power of a (necessarily ample) line bundle on X, then the second Gaussian map $\gamma^2_{\mathcal{O}_X(C)}$ is surjective. Hence we have the following

COROLLARY 4. *([13]) Let X be an abelian surface, let $\mathcal{L}$ be an ample line bundle on X and let $k \geq 7$. Then, for every smooth and irreducible curve $C \in |\mathcal{L}^k|$, the image of second Wahl map*

$$\gamma^2_C : I_2(K_C) \to H^0(K_C^4)$$

is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.

REMARK 4.

1. Using Proposition 3.2 of [19] as in Remark 2, one can prove that for a curve sitting on a very general abelian surface of genus $g > 257$ the second Gaussian map $\gamma^2_{\mathcal{O}_X(C)}$ is surjective, hence the image of second Wahl map $\gamma^2_C : I_2(K_C) \to H^0(K_C^4)$ is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.
2. Assume that $X = E_1 \times E_2$, with E_i elliptic curves, let $p_i : X \to E_i$ be the projection maps, D_i divisors of degree d_i on E_i. As in theorem 3.1 of [10], one shows that if $C \in |p_1^*(D_1) \otimes p_2^*(D_2)|$, and $d_i \geq 7$, then $\gamma^2_{\mathcal{O}_X(C)}$ is surjective, hence also in this case the image of γ^2_C is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.

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EXAMPLES OF CALABI–YAU THREEFOLDS PARAMETRISED BY SHIMURA VARIETIES

Abstract. These are notes from talks of the authors on some explicit examples of families of Calabi–Yau threefolds which are parametrised by a Shimura variety. We briefly review the periods of Calabi–Yau threefolds and we discuss a recent result on Picard–Fuchs equations for threefolds which are hypersurfaces with many automorphisms. Next various examples of families parametrised by Shimura varieties are given. Most of these are due to J.C. Rohde. The examples with an automorphism of order three are given in some detail. We recall that such families do not have maximally unipotent monodromy and that the Shimura varieties in these cases are ball quotients.

Introduction

Calabi–Yau threefolds have been intensively studied in the context of Mirror Symmetry, which originated in theoretical physics. The most basic version of this principle states that given a CY threefold X, there should exist a CY threefold Y, the Mirror of X, with $h^{1,1}(X)=h^{2,1}(Y)$ and $h^{2,1}(X)=h^{1,1}(Y)$. A further requirement is that the variation of Hodge structures on the third cohomology group in the deformations of X is related, in specific way, to the Kähler cone in the second cohomology group of the deformations of Y. There are now quite a few cases where such Mirror pairs have been found and where profound aspects of Mirror Symmetry, like the relation with Gromov–Witten invariants, have been verified.

As CY threefolds are assumed to be projective (or at least Kähler), in case $h^{2,1}(X)=0$ there cannot exist a Mirror of X. Quite a few of such (rigid) threefolds X are known and a modification of Mirror Symmetry, in one specific case, was proposed in [6]. The expected relation between deformations of X and the Kähler cone of Y requires that there exist boundary points in the (complex structure) moduli space of X where the variation of Hodge structures on the H^3 has maximally unipotent monodromy. It was recently pointed out by J.C. Rohde [25] that there do exist families of CY threefolds which do not admit such boundary points. The moduli spaces of the families in question are Shimura varieties. No modification of Mirror Symmetry in these cases is known to us.

Having a base which is a Shimura variety is otherwise a quite desirable property because then one has a very good control over the variation of the Hodge structures. Moreover, the so-called CM points (cf. [4, 24]) will be dense in the moduli space. Physicists expect the field theory on the corresponding Calabi–Yau threefolds to be simpler [19].

In these notes, we give various examples of CY threefolds whose moduli space is a Shimura variety. A very simple example, where the Shimura variety is the moduli space of elliptic curves, is given in Section 2.3. The CY threefolds are of Borcea–

Voisin type and it is easy to see that the family does not have maximally unipotent monodromy.

In Section 3 we discuss various examples, due to J.C. Rohde, of CY threefolds with an automorphism of order three. Their moduli spaces are Shimura varieties which are ball quotients and they do not have maximally unipotent monodromy. We show, in all cases but one, that Rohde's CY threefolds are desingularizations of a quotient of a product of two fixed elliptic curves with another curve of higher genus. These curves of higher genus also have an automorphism of order three. The moduli space of CY threefolds is, locally at least, the moduli space of such pairs (C,β) where C is the higher genus curve and β is its automorphism of order three. This allows one to write down the Picard–Fuchs equation for the variation of Hodge structures explicitly in one case, see Section 3.3. New examples similar to these, but using an automorphism of order four, can be found in [13].

1. Periods of Calabi–Yau threefolds

Good references for CY threefolds and Mirror Symmetry are the overview of M. Gross in [18] and the book [9].

1.1. Calabi–Yau threefolds

In these notes, a Calabi–Yau threefold X is a smooth, three-dimensional (complex) projective variety with trivial canonical bundle

$$\Omega^3_X \cong \mathcal{O}_X, \qquad \text{and} \qquad H^1(X,\mathcal{O}_X) = 0.$$

Using Serre duality one then finds that

$$H^2(\mathcal{O}_X) \cong H^1(\Omega^3_X)^* = H^1(\mathcal{O}_X) = 0 \quad \text{and} \quad H^3(\mathcal{O}_X) \cong H^0(\Omega^3_X)^* \cong \mathbb{C}.$$

Examples of CY threefolds are quintic hypersurfaces in $\mathbb{P}^4$ and complete intersections of two hypersurfaces of degree three in $\mathbb{P}^5$. Many more families of CY threefolds can be found as hypersurfaces in four-dimensional toric varieties [20]. There are obvious restrictions on the topology of X, in fact Hodge theory easily implies that $H^1(X,\mathbb{C}) = 0$ and $\dim H^3(X,\mathbb{C}) \geq 2$. As X is projective, one has $\dim H^2(X,\mathbb{C}) \geq 1$. Poincaré duality implies that $b_i = b_{6-i}$, but no further restrictions on the Betti numbers $b_i := \dim H^i(X,\mathbb{C})$ of X are known.

In analogy with the case of curves, abelian varieties and K3 surfaces, one studies the Hodge structures on the cohomology groups in order to understand these varieties better. For CY threefolds, only H^3 is of interest, as $H^2(X,\mathbb{C}) = H^{1,1}(X)$.

1.2. The polarized Hodge structure on H^3

The Hodge structure on $H^3(X,\mathbb{Z})/\text{torsion}$ is the decomposition of its complexification:

$$H^3(X,\mathbb{C}) = \underbrace{\underbrace{H^{3,0}(X)}_{F^3} \oplus H^{2,1}(X)}_{F^2} \oplus H^{1,2}(X) \oplus H^{0,3}(X), \qquad H^{p,q}(X) = \overline{H^{q,p}}(X).$$

From the F^2 of the Hodge filtration, one recovers $H^3(X,\mathbb{C}) = F^2 \oplus \overline{F^2}$. The intersection form on $H^3(X,\mathbb{Z})$, which factors over $H^3(X,\mathbb{Z})/\text{torsion}$, defines a polarization Q_X on this Hodge structure:

$$Q_X : (H^3(X,\mathbb{Z})/\text{torsion}) \times (H^3(X,\mathbb{Z})/\text{torsion}) \longrightarrow \mathbb{Z},$$

$$Q_X(\theta_1,\theta_2) := \int_X \theta_1 \wedge \theta_2,$$

where we identified $H^3(X,\mathbb{Z})/\text{torsion}$ with the image of $H^3(X,\mathbb{Z})$ in $H^3(X,\mathbb{R}) \cong H^3_{DR}(X)$, the de Rham cohomology group. The polarization Q_X is a symplectic form (so it is non-degenerate, unimodular and alternating). It extends to a Hermitian form $H_X(\theta_1,\theta_2) := iQ_X(\theta_1,\overline{\theta_2})$ ($i := \sqrt{-1} \in \mathbb{C}$) on $H^3(X,\mathbb{C})$ for which the Hodge decomposition is orthogonal:

$$H_X(v,w) = 0 \qquad \text{if} \quad v \in H^{p,q}(X),\, w \in H^{r,s}(X) \quad \text{and} \quad (p,q) \neq (r,s)$$

and which is positive/negative definite on the $H^{p,q}(X)$:

$$H_X := iQ_X \quad \text{is} \quad \begin{cases} >0 & \text{on} \quad H^{3,0}(X), \quad (<0 \quad \text{on} \quad H^{0,3}(X)), \\ <0 & \text{on} \quad H^{2,1}(X), \quad (>0 \quad \text{on} \quad H^{1,2}(X)). \end{cases}$$

1.3. The Period domain

Let $N = b_3$ be the rank of $H^3(X,\mathbb{Z})$ and let Q be a (fixed) symplectic form on $V_{\mathbb{Z}} := \mathbb{Z}^N$. Then we consider the period space $\mathcal{D} = \mathcal{D}_N$ of all polarized weight three Hodge structures on $(V_{\mathbb{Z}},Q)$ of CY type. An element of $\mathcal{D}$ is a decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = V^{3,0} \oplus V^{2,1} \oplus V^{1,2} \oplus V^{0,3}, \qquad V^{p,q} = \overline{V^{q,p}},$$

such that the Hermitian form $H(v,w) := iQ(v,\overline{w})$, with $v,w \in V_{\mathbb{C}}$ and where Q is extended $\mathbb{C}$-linearly, is positive definite on $V^{3,0}, V^{1,2}$ and negative definite on $V^{2,1}, V^{0,3}$. Moreover we require $\dim V^{3,0} = 1$ and we denote $q := \dim V^{2,1}$ (so $N = 2+2q$).

The $(q+1)$-dimensional subspaces W of $V_{\mathbb{C}}$ on which H is positive are parametrised by a $(q+2)(q+1)/2$-dimensional variety (it is isomorphic to the Siegel half space of $(q+1)\times(q+1)$ complex symmetric matrices with positive definite imaginary part). The one dimensional subspaces W_1 of such a subspace W are the points of $\mathbb{P}W$ and thus are parametrised by a q-dimensional variety. Given $W_1 \subset W$, let $W_1^{\perp}$ be the orthogonal complement of W_1 in W. Then we obtain a polarized Hodge structure on $V_{\mathbb{Z}}$ by defining:

$$V^{3,0} := W_1, \qquad V^{2,1} := \overline{W_1^{\perp}}, \qquad V^{1,2} := W_1^{\perp}, \qquad V^{0,3} := \overline{W_1}.$$

Conversely, any polarized Hodge structure on $(V_{\mathbb{Z}}, Q)$ defines a complex line W_1 in a positive $(q+1)$-dimensional subspace W, hence we get

$$\dim \mathcal{D} = q + (q+2)(q+1)/2 = (q^2+5q+2)/2,$$

where $q := \dim V^{2,1}$.

1.4. The Period map

A marking of the CY threefold X is a symplectic isomorphism

$$(H^3(X,\mathbb{Z})/\text{torsion}) \to V_{\mathbb{Z}}.$$

The $\mathbb{C}$-linear extension of this isomorphism maps the Hodge decomposition of $H^3(X,\mathbb{C})$ to a decomposition of $V_{\mathbb{C}}$. In this way we obtain a polarized Hodge structure on $V_{\mathbb{Z}}$. In particular, we get a point $\mathcal{P}(X) \in \mathcal{D}$, the period point of X.

1.5. Deformations of X

An important result, due to Bogomolov, Tian and Todorov, on CY varieties is that the deformations are unobstructed. The first order deformations of a complex variety are parametrised by the cohomology group $H^1(X,\mathcal{T}_X)$, where $\mathcal{T}_X$ is the holomorphic tangent bundle of X. As X is a CY threefold, the cup product pairing $\Omega^1_X \times \Omega^2_X \to \Omega^3_X \cong \mathcal{O}_X$ gives a duality $\mathcal{T}_X \cong (\Omega^1_X)^* \cong \Omega^2_X$ and thus $H^1(X,\mathcal{T}_X) \cong H^1(X,\Omega^2_X) \cong H^{2,1}(X)$. The unobstructedness asserts that there is a neighbourhood B of $0 \in H^1(X,\mathcal{T}_X)$ and there is a family of CY threefolds $\pi : \mathcal{X} \to B$ with fiber $\pi^{-1}(0) = X$, such that the period map $\mathcal{P} : B \to \mathcal{D}$ has an injective differential:

$$(\mathrm{d}\mathcal{P})_0 \,:\, T_0B = H^1(X,\mathcal{T}_X) \cong H^{2,1}(X) \longrightarrow T_{\mathcal{P}(X)}\mathcal{D} \qquad \text{is injective.}$$

Here we used Ehresmann's theorem which asserts that, if B is chosen small enough, there is a diffeomorphism $\phi : \mathcal{X} \to B \times X$ such that $\pi_B\phi = \pi$. As ϕ induces isomorphisms $H^3(X_b,\mathbb{Z}) \cong H^3(X,\mathbb{Z})$ for any $X_b := \pi^{-1}(b)$, we can extend the marking on X to a marking $(H^3(X_b,\mathbb{Z})/\text{torsion}) \to V_{\mathbb{Z}}$ and thus we get the period map $\mathcal{P} : B \to \mathcal{D}$.

Any family of CY threefolds which contains X is locally near X obtained as the pull-back from the family $\mathcal{X} \to B$. Therefore the image of the period map of any family has dimension at most $q = \dim H^{2,1}(X)$ and the image of B has codimension $(q+2)(q+1)/2$ in $\mathcal{D}$. Recent studies of the geometry of $\mathcal{D}$ and these subvarieties are [7] and [21].

1.6. The Picard–Fuchs equation

In this section we will assume for simplicity that $q = \dim H^{2,1}(X) = 1$. With the notation of section 1.5, the diffeomorphism $\phi : \mathcal{X} \to B \times X$ induces an isomorphism of sheaves $R^3\pi_*\mathbb{Z} \overset{\cong}{\to} H^3(X,\mathbb{Z})_B$ on B, where the last sheaf is just the locally constant sheaf defined by the abelian group $H^3(X,\mathbb{Z})$. Using the marking $H^3(X,\mathbb{Z})/\text{torsion} \cong V_{\mathbb{Z}}$ and the Hodge decomposition of $H^3(X_t,\mathbb{C})$ for each deformation X_t of X, we obtain a (trivial) vector bundle $V_{\mathbb{C}} \times B$ over B with holomorphic subbundles

$$\mathcal{F}^3 \subset \mathcal{F}^2 \subset \mathcal{F}^1 \subset \mathcal{F}^0 \cong V_B := V_{\mathbb{C}} \times B, \qquad \mathcal{F}^3_t = H^{3,0}(X_t),$$

where we identify $V_{\mathbb{C}} \times \{t\}$ with $H^3(X_t,\mathbb{C})$.

The period map $\mathcal{P}$ describes the variation of these subbundles inside the trivial bundle $V_{\mathbb{C}} \times B$. Another way to describe this variation is to take a non-vanishing section ω of the rank one bundle $\mathcal{F}^3$, so $\omega(t)$ is a basis of $H^{3,0}(X_t)$ for all $t \in B$. The trivial bundle V_B comes with the Gauss–Manin connection ∇ which maps the horizontal sections $s_v := t \mapsto (v,t)$ to zero, where $v \in V_{\mathbb{C}}$:

$$\nabla = \nabla_{\partial/\partial t} : V_{\mathbb{C}} \times B \longrightarrow V_{\mathbb{C}} \times B.$$

Applying the connection i times to the section ω, we get a section $\nabla^i\omega$. As $\dim V_{\mathbb{C}} = 2 + 2q = 4$, there must be a linear relation, with coefficients $p_i(t)$ which will be holomorphic in t:

$$\mathbf{D}\,\omega = 0, \qquad \mathbf{D} := \sum_{i=0}^{4} p_i(t)\nabla^i.$$

This linear relation is known as the Picard–Fuchs equation.

Instead of considering this rank four bundle with its section ω, one can also choose a basis $\gamma_1,\ldots,\gamma_4$ of $H_3(X,\mathbb{Z})/\text{torsion}$ and define four holomorphic functions $\varphi_i(t) := \int_{\gamma_i} \omega(t)$ on B, where γ_i is identified with a cycle in $H_3(X_t,\mathbb{Z})/\text{torsion}$ using the diffeomorphism ϕ. These four functions are a basis of the solutions of the degree four differential operator $\sum_{i=0}^4 p_i(t)(\mathrm{d}/\mathrm{d}t)^i$ which is also called the Picard–Fuchs equation for the family $\mathcal{X} \to B$.

1.7. An example

The Dwork pencil of quintic threefolds in $\mathbb{P}^4$ is defined by the equation

$$X_t : \qquad X_1^5 + \ldots + X_5^5 - 5tX_1X_2\cdots X_5 = 0.$$

For general $t \in \mathbb{C}$, the variety X_t is a CY threefold with $h^{1,1}(X_t) = 1$ and $q = h^{2,1}(X_t) = 101$. However, there is a finite subgroup $G \cong (\mathbb{Z}/5\mathbb{Z})^3$ acting on $\mathbb{P}^4$ which induces automorphisms on each X_t and the third cohomology group splits under this action:

$$H^3(X_t,\mathbb{Q}) = T_t \oplus S_t,$$

with

$$T_t := H^3(X_t,\mathbb{Q})^G \cong \mathbb{Q}^4, \qquad H^{3,0}(X_t) \subset T_t \otimes_{\mathbb{Q}} \mathbb{C}.$$

Thus the G-invariant part of the cohomology gives a four-dimensional variation of polarized Hodge structures and thus it gives a degree four Picard–Fuchs equation.

In the context of Mirror Symmetry, it was observed that the (singular) quotient variety X_t/G has a resolution of singularities M_t which is a CY threefold, moreover its Hodge numbers are:

$$h^{1,1}(M_t) = 101, \qquad h^{2,1}(M_t) = 1, \qquad M_t := \widetilde{X_t/G}.$$

Note that $h^{p,q}(M_t) = h^{3-p,q}(X_t)$, which is one of the requirements for the (101-dimensional) family of quintic CY threefolds and the one parameter family of M_t's to be Mirror CY families.

The quotient map induces an isomorphism $T_t \cong H^3(M_t, \mathbb{Q})$. In particular, the degree four Picard–Fuchs equation obtained from the variation of the T_t is the Picard–Fuchs equation of the one parameter family of CY threefolds M_t. A spectacular result from Mirror Symmetry is that a certain solution of this Picard–Fuchs equation defines a power series in one variable whose coefficients a_d allow one to compute the Gromov–Witten invariants of a quintic threefold, that is, roughly, the number of rational curves of degree d on a quintic threefold.

In the paper [16], Greene, Plesser and Roan verify that there is an action of the group $H \cong \mathbb{Z}/41\mathbb{Z}$ on $\mathbb{P}^4$ such that each member of the pencil of quintic threefolds

$$Y_t : \qquad X_1X_2^4 + X_2X_3^4 + \ldots + X_5X_1^4 - 5tX_1X_2\cdots X_5 = 0$$

is invariant under H. This leads, as above, to a splitting

$$H^3(Y_t, \mathbb{Q}) = T_t' \oplus S_t',$$

where

$$T_t' := H^3(Y_t, \mathbb{Q})^H \cong \mathbb{Q}^4, \qquad H^{3,0}(Y_t) \subset T_t' \otimes_{\mathbb{Q}} \mathbb{C}.$$

Moreover, the degree four Picard–Fuchs equation defined by the variation of the Hodge structures T_t' is the same as the Picard–Fuchs equation obtained from the variation of the $T_t \cong H^3(M_t, \mathbb{Q})$. In [11] more such examples are given. A possible explanation would be that the CY threefold M_t is birationally isomorphic to a desingularization of Y_t/H.

This is indeed the case. Using results of Shioda, in the recent paper [3] it is shown that there is a commutative diagram, where the arrows are rational maps which are quotients by certain finite groups on suitable Zariski open subsets:

$$\begin{tikzcd}
 & \tilde{X}_{dI,t} \arrow[dl] \arrow[dr] & \\
X_t \arrow[dr] & & Y_t, \arrow[dl] \\
 & M_t &
\end{tikzcd}$$

where

$$\tilde{X}_{dI,t} : \quad X_1^d + \ldots + X_5^d - 5t(X_1X_2\cdots X_5)^{d/5} = 0,$$

and $d = 5^2 \cdot 41 = 1025$. Using the full diagram, one can show that the map $Y_t \to M_t$ has degree 41 and factors over Y_t/H. Thus there is a birational isomorphism between Y_t/H and M_t.

More generally, one can replace Y_t by a family with an equation

$$\sum_{j=1}^{5} \prod_{i=1}^{5} X_i^{a_{ij}} - 5tX_1 \cdots X_5$$

for suitable 5×5 matrices with integer coefficients a_{ij}. This again can be generalized to any number of variables. A further generalization to weighted projective spaces is given in [2].

2. CY threefolds parametrised by Shimura varieties

The period space $\mathcal{D} = \mathcal{D}_N$ parametrises the polarized weight three Hodge structures of CY type on $(V_{\mathbb{Z}} \cong \mathbb{Z}^N, Q)$. Given a CY threefold X and a marking, the unobstructedness of the deformations of X implies that the period points of all deformations of X are the points of a $q = (N-2)/2$-dimensional subvariety B of $\mathcal{D}$.

On the other hand, there are many Hermitian symmetric domains which parametrise Hodge structures of CY type. Such a domain is of the form $G(\mathbb{R})/K$, where $G(\mathbb{R})$ is a real reductive Lie group which is the group of real points of an algebraic group defined over $\mathbb{Q}$, and K is maximal compact subgroup of $G(\mathbb{R})$. Given $(V_{\mathbb{Z}}, Q)$, there is a fixed representation of $G(\mathbb{R})$ on $V_{\mathbb{R}}$ such that the image of $G(\mathbb{R})$ is contained in the symplectic group $Sp(Q,\mathbb{R})$. One considers the homomorphisms of real Lie groups

$$h : S^1 = \{z \in \mathbb{C} : |z| = 1\} \longrightarrow G(\mathbb{R})$$

such that the eigenvalues of $h(z)$ are $z^p\overline{z}^q$ with non-negative integers p,q such that $p+q = 3$. Each such homomorphism gives a Hodge structure of weight three on $V_{\mathbb{Z}}$ by defining $V^{p,q}$ to be the eigenspace with eigenvalue $z^p\overline{z}^q$. The group $G(\mathbb{R})$ acts on the set of such Hodge structures by conjugation $h \mapsto ghg^{-1}$ with $g \in G(\mathbb{R})$. See for example [24], Chapter 1.

Changing the marking corresponds to an action of an element of $\Gamma := Sp(V_{\mathbb{Z}}, Q)$ on $\mathcal{D}$. The moduli space of the Hodge structures of deformation of X is thus the quotient of B by the subgroup Γ_B of Γ which maps B into itself. Not much is known about the subgroups Γ_B, see however [12, 23] for a study in the case $N = 4$. In case B is a Hermitian symmetric domain and Γ_B is an arithmetic subgroup of $G(\mathbb{Q})$ one obtains a Shimura variety $\Gamma_B\backslash B = \Gamma_B\backslash G(\mathbb{R})/K$ which parametrises the deformations of X.

Below we will review various examples from [24]. In general it is not easy to decide if the deformations of a CY threefold are parametrised by a Shimura variety. See [15] for a family of CY threefolds which are not parametrised by a Shimura variety.

2.1. Example

The reference is [4, §3]. Let E_i, $i = 1,2,3$ be elliptic curves and let $\iota_i : E_i \to E_i$ be the inversion $z \mapsto -z$ for the group law on E_i. Let

$$G_4 := \langle \iota_1 \times \iota_2 \times 1_{E_3},\ \iota_1 \times 1_{E_2} \times \iota_3 \rangle \quad \subset \ \mathrm{Aut}(E_1 \times E_2 \times E_3).$$

Then the (singular) variety $(E_1 \times E_2 \times E_3)/G_4$ has a resolution of singularities which is a CY threefold X with $h^{2,1} = 3$ (and $h^{1,1} = 51$). Thus the deformation space of X is three dimensional. Obviously, it contains the CY varieties obtained by deforming the three elliptic curves. Thus the period points of deformations of X are in $B = \mathbb{H}_1^3$, where $\mathbb{H}_1$ is the upper half plane which parametrises elliptic curves. Thus these CY's are parametrised by a Shimura variety.

2.2. Examples of Borcea–Voisin type

Let S be a K3 surface admitting an involution α_S such that $H^{2,0}(S)$ is in the eigenspace of the eigenvalue -1 for the action of α_S^* on $H^2(S,\mathbb{C})$. We will assume moreover that the fixed locus of the involution α_S is made up of k rational curves. The dimension of the family of K3 surfaces admitting an involution acting non trivially on $H^{2,0}$ and fixing k rational curves is $10-k$ and such a family is parametrised by a Shimura variety associated to $SO(2,10-k)$.

Let E be an elliptic curve and let ι be the involution $z \mapsto -z$ on E. The quotient threefold $(S \times E)/(\alpha_S \times \iota)$ admits a desingularization which is a CY threefold X (this construction is called Borcea–Voisin construction). In [29, 5] the Hodge numbers of X are computed:

$$h^{1,1}(X) = 15 + 5k, \qquad h^{2,1}(X) = 11 - k.$$

Hence the dimension of the family of the Calabi–Yau threefolds determined by X is the sum of the dimension of the family of the K3 surfaces with involution and the dimension of the family of elliptic curves. Thus these CY threefolds are parametrised by the product of the Shimura varieties parametrising these two families, see [24, Section 11.3].

Example 2.1 is a particular case of this construction, indeed the desingularization of the quotient $(E_1 \times E_2)/(\iota_1 \times \iota_2)$ is a K3 surface S (in fact, it is a Kummer surface). The automorphism α_S induced on S by $1_E \times \iota$ acts non trivially on $H^{2,0}(S)$ and fixes 8 rational curves. Hence $(S \times E_3)/(\alpha_S \times \iota)$ is birational to $(E_1 \times E_2 \times E_3)/G_4$. For the Shimura varieties, one should remember that the real Lie groups $SO(2,2)^0$ and $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ are isogeneous and thus the Shimura variety associated to $SO(2,2)$ is indeed a quotient of $\mathbb{H}_1 \times \mathbb{H}_1$.

2.3. The easiest case

Another particular case of the Borcea–Voisin construction is obtained by choosing S to be the unique K3 surface with an automorphism α_S which fixes $k = 10$ rational curves. This K3 surface S is well known. It is described, for example, in [26] as the desingularization of the quotient

$$(E_{\sqrt{-1}}\times E_{\sqrt{-1}})/(\gamma_E\times\gamma_E^3),\qquad \text{with}\quad E_{\sqrt{-1}}:=\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$$

and γ_E is the automorphism of $E_{\sqrt{-1}}$ defined by $z\mapsto\sqrt{-1}z$.

The third cohomology group of X is:

$$H^3(X,\mathbb{Q})\cong\left(T_S\otimes H^1(E,\mathbb{Q})\right)^{\alpha_S\times\iota}\cong T_S\otimes H^1(E,\mathbb{Q}),$$

where

$$T_S:=H^2(S,\mathbb{Q})^{\alpha_S=-1}\cong\mathbb{Q}^2.$$

As S, and thus the Hodge structure on T_S, is fixed, the variation of Hodge structures in $H^3(X,\mathbb{Z})$ comes from the variation of Hodge structures on $H^1(E,\mathbb{Z})$. This rank $N=4$ variation is the direct sum of two (identical) rank two deformations, and is parametrised by $\mathbb{H}_1$. In particular, this variation of Hodge structures does not have maximally unipotent monodromy, instead the monodromy operators have 2×2 diagonal blocks in a suitable basis, cf. [25, Example 1]. We will see more examples of variations without maximally unipotent monodromy in Section 3.2.

Any CY threefold from this family is also birationally isomorphic to a double cover of $\mathbb{P}^3$ branched along the union of eight planes As such it appears as entry no. 13 in the table in Section 4.2.5 of the book [22]. To obtain this double cover, one uses that S is a double cover of $\mathbb{P}^2$ branched over six lines. Putting one line "at infinity" in $\mathbb{P}^2$, a birational model of S is (cf. [28, 5.1, 5.2]):

$$S:\qquad s^2=xy(x-1)(y-1)(x-y).$$

An elliptic curve E can be defined by $t^2=u(u-1)(u-\lambda)$ for a suitable $\lambda\in\mathbb{C}$. Hence X, a desingularization of $S\times E$ by the involution which fixes x,y,u and maps $s,t\mapsto -s,-t$, has a birational model defined by

$$X:\qquad w^2=xyu(x-1)(y-1)(u-1)(x-y)(u-\lambda).$$

A suitable coordinate transformation on $\mathbb{P}^3$ will map the branch locus to the one in Meyer's book [22].

2.4. CY-type Hodge structures parametrised by Shimura varieties

There are many Shimura varieties which do parametrise variations of Hodge structures of CY type, but where it is not known if these Hodge structures come from CY threefolds.

For example, let A be an abelian threefold and let $L\in H^2(A,\mathbb{Z})$ be an ample divisor class. Note that A is not a CY since $h^{1,0}=h^{2,0}=3$. One has $H^3(A,\mathbb{Z})=\wedge^3H^1(A,\mathbb{Z})$ and the primitive cohomology

$$H^3(A,\mathbb{Z})_{\mathrm{prim}}\cong H^3(A,\mathbb{Z})/\left(L\wedge H^1(A,\mathbb{Z})\right)\qquad(\cong\mathbb{Z}^{14})$$

is a polarized Hodge structure of CY type with $q = h^{2,1}_{\mathrm{prim}} = 9-3 = 6$. The moduli space $\mathcal{A}_{3,L}$ of polarized abelian threefolds with the same polarization type as (A,L) (these are all deformations of A) is the quotient $\Gamma_L \backslash \mathbb{H}_3$ of the Siegel half space by a discrete subgroup $\Gamma_L \subset Sp(6,\mathbb{Q})$. The image of $\mathbb{H}_3$ in $\mathcal{D}_{14}$ parametrises the polarized Hodge structures of CY type $H^3(A_t,\mathbb{Z})_{\mathrm{prim}}$ for deformations (A_t,L_t) of (A,L), so these Hodge structures are parametrised by a Shimura variety. To the best of our knowledge, it is not known if there exists a family of CY threefolds X_t such that $H^3(A_t,\mathbb{Z})_{\mathrm{prim}} \cong H^3(X_t,\mathbb{Z})/\text{torsion}$ for t in an open, dense, subset of $\mathbb{H}_3$.

A family of polarized CY-type Hodge structures, with $h^{2,1} = 27$, parametrised by the Hermitian symmetric domain associated to the Lie group of type E_7 is defined in [17]. It is not yet known if there is a family of CY threefolds with these Hodge structures.

3. Examples with automorphisms of order three

3.1. Rohde's construction

In the paper [25], J.C. Rohde constructs families of CY threefolds with $q = h^{2,1} = 6-k$, for $0 \le k \le 6$, which are parametrised by a q-dimensional Shimura variety, in this case a ball quotient. They are obtained as the desingularization of the quotient of a product $E \times S$ by an automorphism of order three, where E is a certain elliptic curve and S is a K3 surface which admits an automorphism of order three which fixes k rational curves and $k+3$ isolated points. These K3 surfaces were classified in [1]. A similar construction with an automorphism of order four is discussed in [13], and various examples are given.

Let $\xi \in \mathbb{C}$ be a primitive cube root of unity and consider the elliptic curve

$$E := \mathbb{C}/\mathbb{Z}+\mathbb{Z}\xi, \qquad \mathrm{End}(E) = \mathbb{Z}[\xi].$$

We let α_E be the automorphism of E defined by $z \mapsto \xi z$. A Weierstrass equation of E is $y^2 = x^3 - 1$ and $\alpha_E : (x,y) \mapsto (\xi x, y)$. The automorphism α_E gives the decomposition into eigenspaces, with eigenvalues ξ and $\overline{\xi}$ respectively:

$$H^1(E,\mathbb{C}) = H^{1,0}(E)_{\xi} \oplus H^{0,1}(E)_{\overline{\xi}}.$$

For any integer k with $0 \le k \le 6$, there exist K3 surfaces S with an automorphism of order three α_S such that the second cohomology group splits as:

$$H^2(S,\mathbb{Q}) = T_S \oplus N_S,$$

where

$$N_S := H^2(S,\mathbb{Q})^{\alpha_S} \cong \mathbb{Q}^{8+2k}, \qquad H^{2,0}(S) \subset T_S \otimes \mathbb{C}.$$

As $\dim H^2(S,\mathbb{Q}) = 22$, it follows that $\dim T_S = 14-2k$ and the Hodge numbers of the weight two polarized Hodge structure T_S are $h^{2,0}(T_S) = 1$, $h^{1,1}(T_S) = 12-2k = 2q$. The action of α_S^* on T_S defines a structure of $\mathbb{Q}(\xi)$-vector space on T_S. The eigenspaces for this action are

$$T_S \otimes \mathbb{C} = \underbrace{T^{2,0}_{S,\bar{\xi}} \oplus T^{1,1}_{S,\bar{\xi}}}_{T_{S,\bar{\xi}}} \oplus \underbrace{T^{1,1}_{S,\xi} \oplus T^{0,2}_{S,\xi}}_{T_{S,\xi}},$$

and

$$\dim T^{1,1}_{S,\bar{\xi}} = \dim T^{1,1}_{S,\xi} = 6 - k = q.$$

The moduli space of such K3 surfaces is q-dimensional, and it is a quotient of the q-ball in $\mathbb{C}^q$, see Section 3.5, and it is a Shimura variety.

The weight three polarized rational Hodge substructure of $\alpha := \alpha_S \times \alpha_E$-invariants in the tensor product $H^2(S,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})$ is then of CY type. Rohde shows that it is isomorphic to the third cohomology group of a CY threefold X_S which is a desingularization of the (singular) quotient variety $(S \times E)/(\alpha_S \times \alpha_E)$:

$$H^3(X_S,\mathbb{Q}) \cong \left(H^2(S,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})\right)^{\alpha} = \left(T_S \otimes H^1(E,\mathbb{Q})\right)^{\alpha_S \times \alpha_E}.$$

Here it is important that the fixed point locus of the automorphism α_S on S consists of (smooth) rational curves and isolated points.

The CY threefold X_S still has an automorphism α_{X_S} of order three which is induced by $1_S \times \alpha_E$ (or, equivalently $\alpha_S^{-1} \times 1_E$). As the eigenspaces of α_E on $H^1(E,\mathbb{Q})$ are $H^{1,0}(E)$ and $H^{0,1}(E)$ we obtain the decomposition into α_{X_S}-eigenspaces:

$$H^3(X_S,\mathbb{C}) \cong \left(T_{S,\bar{\xi}} \otimes H^{1,0}(E)_{\xi}\right) \oplus \left(T_{S,\xi} \otimes H^{0,1}(E)_{\bar{\xi}}\right) = F^2 \oplus \overline{F^2}$$

where the last equality follows by inspection of the Hodge decomposition of T_S:

$$H^{3,0}(X_S) \cong T^{2,0}_{S,\bar{\xi}} \otimes H^{1,0}(E)_{\xi} \quad \subset \quad T_{S,\bar{\xi}} \otimes H^{1,0}(E)_{\xi},$$

and similarly

$$H^{2,1}(X_S) \cong T^{1,1}_{S,\bar{\xi}} \otimes H^{1,0}(E)_{\xi} \qquad \text{and thus} \quad \dim H^{2,1}(X_S) = q.$$

As the moduli of S already provide a q-dimensional deformation space of X_S, one finds that all the deformations of X_S are of this type. Therefore these CY threefolds are parametrised by the same Shimura variety as the K3 surfaces S.

3.2. No maximal unipotent monodromy

A peculiar feature of these families of CY threefolds is that they do not have a large complex structure limit. In other words, their Picard–Fuchs equations do not have singular points with maximally unipotent monodromy. To see why, recall that $F^2 = H^3(X_S,\mathbb{C})_{\xi}$ and $\overline{F^2} = H^3(X_S,\mathbb{C})_{\bar{\xi}}$ are the eigenspaces of α_{X_S}. The non-vanishing section ω of $F^3 \subset F^2 = F^2_{\xi}$ is always in the ξ-eigenspace, which has dimension $1+q$.

Therefore its derivatives under the Gauss–Manin connection remain in this eigenspace. Instead of a degree $2(1+q)$ Picard–Fuchs equation one now finds an equation of degree $1+q$. This implies that the “standard” recipe for Mirror Symmetry cannot be applied to these families. In the case $q=1$, this equation was given explicitly in [14], see also the next section. We already saw another example of this in Section 2.3.

3.3. The case $q=1$

We recall our explicit description from [14] of the K3 surfaces from Section 3.1 in the case the associated CY threefolds have $q=h^{2,1}=1$.

One starts with a polynomial $f=gh^2\in\mathbb{C}[t]$ with g,h of degree two such that f has four distinct zeroes, so up to the action of $\mathrm{Aut}(\mathbb{P}^1)$ we have only one parameter. The K3 surface S_f has an elliptic fibration $\pi: S_f\to\mathbb{P}^1$ with a section. Its Weierstrass model is:

$$S_t:\quad Y^2=X^3+f(t)^2,\qquad f=gh^2,\quad \deg(g)=\deg(h)=2,$$

where t is the coordinate on $\mathbb{P}^1$. This surface has an automorphism of order three

$$\alpha_f:\ S_f\longrightarrow S_f,\qquad (X,Y,t)\longmapsto(\bar{\xi}X,Y,t)$$

which does act as $\bar{\xi}$ on $H^{2,0}(S_f)=\mathbb{C}\mathrm{d}t\wedge\mathrm{d}X/Y$. Explicit computations show that

$$H^2(S_f,\mathbb{Q})\ =\ T_f\oplus N_f,$$

where

$$N_f\ =H^2(S_f,\mathbb{Q})^{\alpha_f}\cong\mathbb{Q}^{18},\qquad H^{2,0}(S_f)\subset T_f\otimes\mathbb{C},$$

and that α_f fixes only five rational curves and eight isolated points. The complexification of T_f splits into four one-dimensional spaces:

$$T_f\otimes\mathbb{C}\ =\ T^{2,0}_{f,\bar{\xi}}\oplus T^{1,1}_{f,\bar{\xi}}\oplus T^{1,1}_{f,\xi}\oplus T^{0,2}_{f,\xi}.$$

Rohde’s construction now produces a CY threefold X_f, the desingularization of the quotient of $S_f\times E$ by the automorphism $\alpha=\alpha_f\times\alpha_E$ and

$$\begin{aligned}H^3(X_f,\mathbb{Q})\ &\cong\ (T_f\times H^1(E,\mathbb{Q}))^{\alpha}\\ &=\ \big(T^{2,0}_{f,\bar{\xi}}\oplus T^{1,1}_{f,\bar{\xi}}\big)\otimes H^{1,0}(E)_{\xi}\oplus\big(T^{1,1}_{f,\xi}\oplus T^{0,2}_{f,\xi}\big)\otimes H^{0,1}(E)_{\bar{\xi}}.\end{aligned}$$

The Hodge structures T_f can be understood better by observing that all the smooth fibers of the elliptic fibration $\pi: S_f\to\mathbb{P}^1$ are isomorphic (they are elliptic curves with j-invariant 0, so are isomorphic to E). Thus the elliptic fibration is isotrivial and becomes birationally isomorphic to a product after a base change. For this, we define a curve

$$C_f:\ v^3\ =\ f(t)$$

which is a 3:1 cyclic cover of $\mathbb{P}^1$ with covering automorphism

$$\beta_f : C_f \to C_f, \qquad (t,v) \longmapsto (t,\xi v).$$

Substituting $f = v^3$ in the Weierstrass equation of S_f, one finds the birational isomorphism:

$$\begin{aligned} C_f \times E &\longrightarrow S_f \approx (C_f \times E)/(\beta_f \times \alpha_E), \\ ((t,v),(x,y)) &\longmapsto (X,Y,t) = (v^2x, v^3y, t). \end{aligned}$$

The automorphism α_f on S_f is induced by α_E. This leads to an isomorphism of Hodge structures:

$$T_f \cong \left(H^1(C_f,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})\right)^{\beta_f \times \alpha_E},$$

which implies another isomorphism of Hodge structures:

$$H^3(X_f,\mathbb{Q}) \cong \left(H^1(C_f,\mathbb{Q}) \otimes H^1(E,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})\right)^H,$$

where $H \cong (\mathbb{Z}/3\mathbb{Z})^2$ is generated by the automorphisms $\beta_f \times \alpha_E \times 1_E$ and $1_{C_f} \times \alpha_E \times \alpha_E^{-1}$ of $C_f \times E \times E$. This shows that the variation of the Hodge structures $H^3(X_f,\mathbb{Q})$ is entirely coming from the variation of the Hodge structures of the curves C_f. Note that

$$H^{3,0}(X_f) \cong H^{1,0}(C_f,\mathbb{Q})_{\overline{\xi}} \otimes H^1(E,\mathbb{Q})_{\xi} \otimes H^1(E,\mathbb{Q})_{\xi}.$$

The Picard–Fuchs equations for the variation of Hodge structures of the curves C_f is explicitly given in [14]. One can parametrise $\mathbb{P}^1$ in such a way that $g(t) = t(t-1)$ and $h(t) = (t-\lambda)$ (and the other zero of h is at ∞), thus $C_f \cong C_\lambda$ with defining equation $v^3 = t(t-1)(t-\lambda)^2$. The holomorphic one forms on this curve are $\mathrm{d}t/v$ and $(t-\lambda)\mathrm{d}t/v^2$, note that they have distinct eigenvalues $\overline{\xi}, \xi$ for the automorphism β_f. The Picard–Fuchs equation for $\eta := \mathrm{d}t/v \in H^1(C_f,\mathbb{Q})_{\overline{\xi}}$ turns out to be:

$$\left(\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + (1-2\lambda)\frac{\partial}{\partial\lambda} - \frac{2}{3}\right)\eta = 0.$$

This is also the Picard–Fuchs equation for the holomorphic three form on the corresponding family of CY threefolds.

Rohde computes the Hodge numbers of these CY threefolds X_f and finds:

$$\dim H^{1,1}(X_f) = 73, \qquad \dim H^{2,1}(X_f) = 1.$$

Any CY threefold Y from the Mirror family, if it exists, should thus have $h^{1,1}(Y) = 1$ and $h^{2,1}(Y) = 73$. At least three families of CY threefolds with these Hodge numbers are known: the complete intersections of type $(3,3)$ in $\mathbb{P}^5$, $(2,2,3)$ in $\mathbb{P}^6$ and $(4,4)$ in the weighted projective space $\mathbb{P}^5(1,1,1,1,2,2)$. But in these cases the Mirror families are known and they have maximally unipotent monodromy (cf. [8]), hence they cannot be the Mirrors of the family of the X_f.

3.4. The case $q > 1$

In case $q \leq 5$, we again find that the K3 surface S has an isotrivial fibration with smooth fibers isomorphic to E, but we could not find such a fibration in case $q = 6$. The CY threefold X_S is then again a desingularization of a quotient of the product of a curve C with two copies of the fixed elliptic curve E. The variation of Hodge structures of the X_S is obtained from the deformations of C.

For $q \leq 3$, we consider the surface (cf. [14])

$$S_f : y^2 = x^3 + f(t)^2, \;\; f = gh^2, \qquad \deg(f) = 6,$$

such that g and h have no common zeros and no multiple zeros. The curve $C_f : \; v^3 = f(t)$ has the automorphism $\beta_f : (t,v) \to (t,\xi v)$. As in 3.3, Rohde's CY threefold X_f is the desingularization of $(C_f \times E \times E)/H$, where $H = \langle \beta_f \times \alpha_E \times 1_E, 1_{C_f} \times \alpha_E \times \alpha_E^{-1} \rangle$ (see [14, Remark 1.3]). The Hodge numbers of X_f and the genus $g(C_f)$ of C_f are as follows:

$\deg(g)$	$\deg(h)$	$g(C_f)$	$q = h^{2,1}(X_f)$	$h^{1,1}(X_f)$
6	0	4	3	51
4	1	3	2	62
2	2	2	1	73
0	3	1	0	84

The last line corresponds to a rigid CY threefold X_f where $C_f \simeq E$, and X_f is the desingularization of the quotient $E \times E \times E$ by $\langle \alpha_E^{-1} \times \alpha_E \times 1_E, \; 1_E \times \alpha_E \times \alpha_E^{-1} \rangle$. In this case, the K3 surface S_f is described in [26].

In case $q = 4$, we consider the curve

$$C_l : \;\; v^6 = l(t) \qquad \deg(l) = 12$$

such that $l(t)$ has 5 double zeros. It admits the automorphism $\beta_l : (t,v) \mapsto (t,\xi v)$. The quotient $(C_l \times E)/(\beta_l \times \alpha_E)$ has a desingularization S_l which is a K3 surface having an elliptic fibration with Weierstrass equation $Y^2 = X^3 + l(t)$, where $X := v^2 x$, $Y := v^3 y$. The surface S_l admits an automorphism α_l of order 3 induced by α_E. The fixed locus of α_f consists of 2 rational curves and 5 points. Applying Rohde's construction to the K3 surface S_l one obtains a CY threefold X such that $h^{2,1}(X) = q = 4$ and $h^{1,1}(X) = 40$.

In case $q = 5$, one needs a K3 surface S with an automorphism α_S of order 3 which fixes one rational curve and 4 points (cf. [25]). In [1] a projective model of such a surface is given: it is a (singular) complete intersection in $\mathbb{P}^4$ with equations

$$\begin{cases} F_2(x_0,\ldots,x_3) = 0, \\ G_3(x_0,\ldots,x_3) = x_4^3, \end{cases}$$

where F_2 and G_3 are homogeneous polynomials of degree 2 and 3 respectively. Moreover, the curve $V(F_2)\cap V(G_3)$ has 4 singular points of type A_1. The surface S is clearly a triple cover of the quadric defined $F_2 = 0$ in $\mathbb{P}^3$ branched over the curve which is the intersection of this quadric with the cubic surface defined by $G_3 = 0$ in $\mathbb{P}^3$. The inverse image in S of a line in a ruling of the quadric in $\mathbb{P}^3$ is an elliptic curve with a covering automorphism of order three which fixes the ramification points. Hence such an elliptic curve is isomorphic to E. Thus S admits an isotrivial fibration (in general without section) with general fiber isomorphic to E. In this case Rohde's CY threefold has $h^{1,1}(X) = 29$.

3.5. The complex ball

We briefly recall why the CY-type Hodge structures $H^3(X_S,\mathbb{Z})$ are parametrised by a complex q-ball. More generally, with the notation from Section 1.3, consider polarized weight three Hodge structures on $(V_\mathbb{Z} \cong \mathbb{Z}^{2(1+q)}, Q)$ of CY type which, moreover, admit an automorphism of order three ϕ:

$$\phi : V_\mathbb{Z} \longrightarrow V_\mathbb{Z}, \qquad Q(\phi x, \phi y) = Q(x,y), \qquad \phi_\mathbb{C}(V^{p,q}) = V^{p,q}, \quad \phi^3 = 1_{V_\mathbb{Z}}.$$

Then we have a decomposition of $V_\mathbb{C}$ into ϕ-eigenspaces, and we assume, as in the examples above, that the eigenspace of ϕ with eigenvalue ξ is exactly F^2, so $F^2 = F^2_\xi$. Then the $V^{p,q}$ are also ϕ-eigenspaces:

$$V_\mathbb{C} = V_\xi \oplus V_{\bar\xi} = V_\xi^{3,0} \oplus V_\xi^{2,1} \oplus V_{\bar\xi}^{2,1} \oplus V_{\bar\xi}^{0,3}.$$

In particular, the subspace F_2, being an eigenspace of the fixed automorphism ϕ of $V_\mathbb{Z}$, is now fixed in $V_\mathbb{C}$. It remains to find the moduli of $V^{3,0}$ inside $F^2 = V^{3,0} \oplus V^{2,1}$. Recall the Hermitian form H on $V_\mathbb{C}$ which is positive definite on $V^{3,0}$ and negative definite on $V^{2,1}$. These two subspaces are perpendicular for H. Thus the unitary group of $H_{|F^2}$ is isomorphic to the group $U(1,q)$. It is well-known that this group acts transitively on the orthogonal decompositions $F^2 = W \oplus W^\perp$ with $H_{|W} > 0$ (and thus $H_{|W^\perp} < 0$). The stabiliser of a given decomposition is the subgroup $U(1)\times U(q)$, hence the moduli space of these decompositions is the Hermitian symmetric domain

$$U(1,q)/(U(1)\times U(q)) \cong \mathbb{B}^q = \{w \in \mathbb{C}^q : \|w\| < 1\}.$$

It is easy to check that these decompositions correspond to the Hodge structures under consideration, hence the q-ball is also the moduli space of these CY-type Hodge structures.

More explicitly, the Hermitian form H has signature $(1+, q-)$ on the complex subspace $F^2 \cong \mathbb{C}^{1+q}$. Thus F^2 has a basis on which we have $H(z,z) = |z_0|^2 - \sum_{j=1}^n |z_j|^2$. In a Hodge decomposition, we must have $V^{3,0} = \mathbb{C}w'$ for a non-zero $w' = (w_0, w_1, \ldots, w_q) \in F^2$ such that $H(w',w') > 0$, that is, $|w_0|^2 > \sum_{j=1}^n |w_j|^2$. In particular, $w_0 \neq 0$ and so we may assume that $w_0 = 1$. Then w' is determined by the point $w := (w_1, \ldots, w_q) \in \mathbb{C}^q$ with $\sum_{j=1}^n |w_j|^2 < 1$, that is, a point of the q-ball. Conversely, given $w \in \mathbb{B}^q$, let $w' = (1,w)$ and define $V^{3,0} = \mathbb{C}w'$, $V^{2,1} = (V^{3,0})^{\perp}$, the orthogonal complement, w.r.t. H, in F^2 of $V^{3,0}$ and define $V^{1,2}, V^{0,3}$ using $V^{p,q} = \overline{V^{q,p}}$. One easily checks that this gives a polarized Hodge structure on $(V_{\mathbb{Z}}, Q)$ which admits the automorphism ϕ.

As we observed before in Sections 3.3, 3.4, the ball also parametrises families of curves, like the C_f, and K3 surfaces, like the S_f. Equivalently, it also parametrises certain Hodge structures of weight one and two. The relation between these Hodge structures is given by the "half twist" construction, see [27, 10].

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ON SOME LATTICE COMPUTATIONS RELATED TO MODULI PROBLEMS

Abstract. The method used in [6] to prove that most moduli spaces of K3 surfaces are of general type leads to a combinatorial problem about the possible number of roots orthogonal to a vector of given length in E_8. A similar problem arises for E_7 in [8]. Both were solved partly by computer methods. We use an improved computation and find one further case, omitted from [6]: the moduli space $\mathcal{F}_{2d}$ of K3 surfaces with polarisation of degree $2d$ is also of general type for $d = 52$. We also apply this method to some related problems. In Appendix A, V. Gritsenko shows how to arrive at the case $d = 52$ and some others directly.

Introduction

Many moduli spaces in algebraic geometry can be described as locally symmetric varieties, i.e. quotients of a Hermitian symmetric domain $\mathcal{D}$ by an arithmetic group Γ. One method of understanding the birational geometry of such quotients is to use modular forms for Γ to give information about differential forms on $\Gamma\backslash\mathcal{D}$. In [6] this method was used to prove that the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$ is of general type in all but a few cases. The method works if there exists a modular form of sufficiently low weight with sufficiently large divisor. In [6], and again in [8] where a similar method was applied to certain moduli of polarised hyperkähler manifolds, the required modular form is constructed by quasi-pullback of the Borcherds form Φ_{12}.

A suitable quasi-pullback exists if a combinatorial condition is satisfied: there should exist a vector l in the root lattice E_8 (or E_7 in the hyperkähler case) of square $2d$, orthogonal to very few roots. This is evidently the case if d is large, but for small d the search for such an l invites the use of a computer. This was done in both [6] and [8] by a randomised search, relying on the large Weyl group to ensure that in practice no cases would be missed.

Here we present an exhaustive search carried out by the first author. For the hyperkähler case the exhaustive search confirmed the results of the earlier randomised search, but in the K3 case one previously overlooked value of d with a suitable vector was found, namely $d = 52$. In fact it turned out that the randomised search had indeed found this value, and the omission of the case $d = 52$ from [6] happened because the output had been interpreted incorrectly (by GKS).

Nevertheless the following result is true and has not previously appeared in the literature.

THEOREM 1. *The moduli space $\mathcal{F}_{2\cdot 52}$ of* K3 *surfaces with polarisation of degree* 104 *is of general type.*

combinatorial problem is and how it arises, and give some more general combinatorial problems of the same nature. In Section 2 we describe the theoretical and computational methods used to solve it, along with some other results obtained in the same way. In Appendix A, Valery Gritsenko explains how the case $d = 52$ could have been foreseen without the help of a computer. Some of the relevant computer code is given in Appendix B.

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1. Combinatorial problems and moduli

In this section we first give a list of combinatorial questions and then explain the geometry that originally motivated them. First we fix some terminology. We say that L is a *lattice of signature* (a,b) if $L \cong \mathbb{Z}^{a+b}$ and we fix a bilinear form $(\,,\,)\colon L\times L \to \mathbb{Z}$ of signature (a,b). If $x \in L$ we refer to (x,x) as x^2 and call it the *length* of x. If the length of x is 2 then x is called a *root*. If the roots of L generate L as an abelian group then L is called a *root lattice*. A lattice L is *unimodular* if it is equal to its dual $L^\vee = \mathrm{Hom}(L,\mathbb{Z}) \supseteq L$. We do not assume that L is always unimodular but for simplicity we do assume that L is *even*, i.e. that x^2 is always an even integer.

E_8 denotes the unique even unimodular positive-definite lattice of rank 8, i.e. with signature $(8,0)$: this is the sign convention of [3] and is also used in [6]. If $n \in 2\mathbb{Z}$ then $\langle n\rangle$ is the rank 1 lattice spanned by a vector of length n, and U denotes the integral hyperbolic plane $\mathbb{Z}e+\mathbb{Z}f$ with $e^2 = f^2 = 0$ and $(e,f) = 1$. The symbol $\oplus$ denotes the orthogonal direct sum of lattices. If Λ is a lattice and $n \in \mathbb{Z}$, then $\Lambda(n)$ denotes the same lattice with the quadratic form multiplied by n. In particular, $E_8(-1)$ is the negative-definite even unimodular lattice of rank 8.

1.1. Combinatorial problems

Let Λ be a root lattice (usually it will be E_8 or E_7) and denote by $R(\Lambda)$ the set of its roots, i.e. $R(\Lambda) = \{r \in \Lambda \mid r^2 = 2\}$. The combinatorial questions arising in [6] and [8] are special cases of the following.

QUESTION 1. Given integers $p > q \geq 0$, what are the values of d for which every vector of length $2d$ that is orthogonal to at least $2q$ roots is orthogonal to at least $2p$ roots?

More generally we may ask about all possibilities.

QUESTION 2. Given an even natural number $2d$, what are the possible numbers of roots orthogonal to a vector of length $2d$?

If $l \in \Lambda$ we denote by $R(l^\perp)$ the system of roots of Λ orthogonal to l. We denote the answer to Question 2 by $P(\Lambda, d)$: that is

$$P(\Lambda, d) := \{m \in \mathbb{Z} \mid \exists l \in \Lambda\ l^2 = 2d,\ \#R(l^\perp) = m\}. \tag{1}$$

Thus $P(\Lambda, d)$ is a finite set of even non-negative integers. We call this the *root type* of the non-negative even integer $2d$ for the lattice Λ

There are some immediate restrictions on what the root type can be: for example, if $\Lambda = E_8$ then the largest m that can occur is 126, when $R(l^\perp) \cong E_7$; but in that case $l \in (E_7)^\perp_{E_8} \cong A_1$, so d must be a square.

Especially for $\Lambda = E_8$, the value of $m_0(d) = \min P(E_8, d)$ is of interest as it determines the lowest weight of modular form obtained by quasi-pullback (see Equation (2) below). If $m_0(d) = 0$ then this form will not be a cusp form, so the value of $m_1(d) = \min P(E_8, d) \cap \mathbb{N}$ is also significant. We should also like to know whether this form is unique. So we also have the following questions.

QUESTION 3. For given d and Λ, how can we compute $m_0(d)$?

QUESTION 4. For given m, what is the smallest value $d(m)$ of d for which $m_1(d) \leq m$?

If in Question 4 we replace m_1 by m_0, then the case $m = 0$ asks for the length of shortest vectors in the interior of a Weyl chamber: these are the Weyl vectors, which are well known.

If $m \in P(\Lambda, d)$ there is a further natural refinement.

QUESTION 5. How many Weyl group orbits of vectors l with $l^2 = 2d$ and $\#R(l^\perp) = m$ are there?

Some values of m are of particular interest for geometric reasons: for instance, if $14 \in P(E_8, d)$ then quasi-pullback of Φ_{12} gives a canonical form on $\mathcal{F}_{2d}$ (see Section 1.2 below). This leads us to the following variant of Question 1.

QUESTION 6. For given m and Λ, what are the values of d such that $m \in P(\Lambda, 2d)$?

We can compute the answers to some cases of these questions by the methods described in Section 2.

1.2. Moduli

The following construction describes several moduli spaces in algebraic geometry, including the moduli of polarised K3 surfaces.

Let L be an even lattice of signature $(2, n)$. The Hermitian symmetric domain associated with L is $\mathcal{D}_L$, one of the two connected components of

$$\mathcal{D}_L \cup \overline{\mathcal{D}}_L = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid w^2 = 0,\ (w, \overline{w}) > 0\}.$$

The group $\mathrm{O}(L)$ of isometries of L acts on this union and we denote by $\mathrm{O}^+(L)$ the index 2 subgroup preserving $\mathcal{D}_L$. The action is discontinuous, with finite stabilisers, so if Γ is any finite index subgroup of $\mathrm{O}^+(L)$ then

$$\mathcal{F}_L(\Gamma) := \Gamma \backslash \mathcal{D}_L$$

is a complex analytic space. In fact it is a quasi-projective variety, having a minimal projective compactification, the Baily-Borel compactification $\mathcal{F}_L(\Gamma)^*$, obtained by adding finitely many curves (called 1-dimensional cusps) meeting at finitely many points (0-dimensional cusps). It is often preferable to work with a toroidal compactification $\overline{\mathcal{F}_L(\Gamma)}$, which is a modification of $\mathcal{F}_L(\Gamma)^*$ depending on some combinatorial choices at the 0-dimensional cusps.

A modular form for Γ of weight k and character $\chi \colon \Gamma \to \mathbb{C}^*$ is a holomorphic function F on the affine cone $\mathcal{D}_L^\bullet \subset L \otimes \mathbb{C}$ such that

$$F(tZ) = t^{-k} F(Z) \ \ \forall t \in \mathbb{C}^* \quad \text{and} \quad F(gZ) = \chi(g) F(Z) \ \ \forall g \in \Gamma.$$

F is a cusp form if it vanishes at every cusp. For the cases we shall consider the only possible characters are 1 and $\det(g)$, and the order of vanishing at a cusp is an integer: see [7].

The aim of [6] is to show that the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$ is of general type for most values of $d \in \mathbb{N}$. Using the Torelli theorem for K3 surfaces one can show that

$$\mathcal{F}_{2d} = \mathcal{F}_{L_{2d}}(\widetilde{\mathrm{O}}^+(L_{2d})),$$

where $\widetilde{\mathrm{O}}^+(L)$ is the finite index subgroup of $\mathrm{O}^+(L)$ that acts trivially on the discriminant group $L^\vee/L$ and

$$L_{2d} := 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle.$$

Modular forms of suitable weight can be interpreted as differential forms on the moduli space provided that they have sufficiently large divisor. Therefore, to prove that the moduli space is of general type it is enough to give a sufficient supply of such modular forms. There are several technical difficulties here, one of which is the presence of singularities. A sufficient condition, however, was given in [6].

THEOREM 2. *Suppose that $n \geq 9$ and that there exists a nonzero cusp form F_a of weight $a < n$ and character $\chi \equiv 1$ or $\chi(g) = \det(g)$, vanishing along any divisor $\mathcal{H} \subset \mathcal{D}_L$ fixed by reflections in Γ. Then $\mathcal{F}_L(\Gamma)$ is of general type.*

The form F_a is then used to give many forms of high weight with sufficiently large divisor, of the form $F = F_a^k F_{(n-a)k}$, and these in turn give pluricanonical forms on a smooth model of $\overline{\mathcal{F}_L(\Gamma)}$.

To apply this in specific cases such as $\mathcal{F}_{2d}$ one must therefore construct F_a. The method used in [6] to do this is quasi-pullback of the Borcherds form Φ_{12}. This construction first appeared in [2]. The Borcherds form itself was constructed in [1] by means of a product expansion, whereby its divisor is evident. It is a modular form (not a cusp form) of weight 12 and character det for the group $\mathrm{O}^+(II_{2,26})$. The lattice $II_{2,26}$ of signature $(2,26)$ is $2U \oplus N(-1)$, where N is any one of the 24 Niemeier lattices, positive definite unimodular lattices of rank 24: see [4]. For our purposes the correct choice of N is $3E_8$. A choice of a (not necessarily primitive) vector $l \in E_8$ of length $2d$ gives an embedding

$$L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle \hookrightarrow II_{2,26} = 2U \oplus 3E_8(-1)$$

which in turn gives an embedding

$$\mathcal{D}^\bullet_{L_{2d}} \hookrightarrow \mathcal{D}^\bullet_{II_{2,26}}.$$

Denote the images of these embeddings by $L_{2d}[l]$ and $\mathcal{D}^\bullet[l]$ respectively.

If $r \in L$ is a root it determines a Heegner divisor $\mathcal{H}^\bullet_r \subset \mathcal{D}^\bullet_L$, given by the equation $(Z,r) = 0$. The Borcherds form vanishes (to order 1) along all the Heegner divisors for $L = II_{2,26}$ and in particular its restriction to $\mathcal{D}^\bullet[l]$ vanishes, as needed to apply Theorem 2. However, $\Phi_{12}|_{\mathcal{D}^\bullet[l]}$ may well be zero, since if r is a root of $II_{2,26}$ orthogonal to $L_{2d}[l]$ then $\mathcal{D}^\bullet[l] \subset \mathcal{H}^\bullet_r$.

Instead we take the quasi-pullback, simply dividing by the equation of each such $\mathcal{H}^\bullet_r$, noting that $\mathcal{H}^\bullet_{-r} = \mathcal{H}^\bullet_r$. We put

$$R_l = \{r \in R(II_{2,26}) \mid (r, L_{2d}[l]) = 0\} \cong \{r \in R(E_8) \mid (r,l) = 0\}$$

and define the quasi-pullback to be

$$F[l] = \left.\frac{\Phi_{12}}{\prod_{\pm r \in R_l}(r,Z)}\right|_{\mathcal{D}^\bullet[l]}. \tag{2}$$

This is a nonzero modular form, and one can show that it is a cusp form provided $R_l \neq \emptyset$. It vanishes along all the Heegner divisors fixed by reflections in $\mathrm{O}^+(L_{2d})$.

The weight, however, goes up by 1 every time we divide, so the weight of $F[l]$ is $12 + \frac{1}{2}\#R_l$. We can therefore show that $\mathcal{F}_{2d}$ is of general type if we can find an $l \in E_8$ of length $2d$ with $2 \leq \#R_l < 2(n-12) = 14$. Moreover, if we can find a cusp form of weight precisely $n = 19$ then, by a result of Freitag [5], $\mathcal{F}_{2d}$ has $p_g > 0$ and in particular is not uniruled.

This leads us to Question 1, with $q = 1$ and $p = 7$ or $p = 8$, for $\Lambda = E_8$. In [8], similar considerations about the moduli of some hyperkähler manifolds with a certain type of polarisation lead to Question 1 with $q = 1$ and $p = 6$ or $p = 7$, for $\Lambda = E_7$.

2. Solving the combinatorial problems

The specific combinatorial problems encountered in [6] and [8] can be solved in principle by first bounding d. It is clear that for sufficiently large d an l will exist orthogonal to a number of roots in the required range: indeed, for sufficiently large d we can find l orthogonal to exactly two roots. An explicit bound, followed by a finite calculation, will solve the problem. Neither is entirely straightforward, though. In [6] a counting argument is used to show that an $l \in E_8$ with $l^2 = 2d$, orthogonal to at least two and at most 12 roots, exists (and therefore $\mathcal{F}_{2d}$ is of general type) unless

$$28N_{E_6}(2d) + 63N_{D_6}(2d) \geq 4N_{E_7}(2d), \tag{3}$$

where $N_L(2d)$ is the number of ways of representing $2d$ by the quadratic form L. The inequality (3) certainly fails for large d, but to obtain an effective bound on d one must bound $N_{E_6}(2d)$ and $N_{D_6}(2d)$ from above and $N_{E_7}(2d)$ from below by explicit functions. This is a non-trivial problem in analytic number theory but it can be done, and after some refinements it gives a reasonable bound of around $d = 150$. It would be possible to resort to direct computation at that point, but there is no need yet. Some integers in that range are excluded from the list of possibly non-general type polarisations because the inequality (3) (or another similar inequality) in fact fails. Others can be excluded by inspection, actually producing a vector l by guessing the root system $R(l^{\perp}_{E_8})$. The root systems used in this way in [6] were $4A_1$, $2A_1 \oplus A_2$, A_3 and $A_1 \oplus A_2$. The root systems $3A_1 \oplus A_2$ and $2A_2$ were not tried: see Appendix A.

In [8] a similar procedure was used, although there is an extra difficulty caused by the opposite parity of the rank: working in E_7, one needs to estimate $N_R(2d)$ from above for some odd-rank root systems R, and this problem is not so well studied as in the even rank case.

In either case, eventually one is left with a residual list of values of d for which the problem has not been settled. In [6] it consists of most integers between 15 and 60 (for very small d the moduli space is known to be unirational). The residual problem in the hyperkähler case considered in [8] is much smaller.

Now, if we want to be (reasonably) sure that no cases have been missed, we do need a computer. Moreover, the methods we now use to solve this problem can also be used to give answers to question such as those posed in Section 1.1.

2.1. Algorithms

We begin by representing E_8 in the usual way, as the set of points $l = (l_1, \ldots, l_8) \in \mathbb{R}^8$ such that the l_i are either all integers or all strict half-integers (i.e. either $l_i \in \mathbb{Z}$ for all i or $2l_i$ is an odd integer for all i) and $\sum l_i \in 2\mathbb{Z}$, with the standard Euclidean quadratic form on $\mathbb{R}^8$.

We need a very rough upper bound on $N_{E_8}(2d)$, because we want to know whether $N_{E_8}(2d)$ is small enough to allow a brute-force search for $l \in E_8$ with $l^2 = 2d$ having $2 \leq \#R(l^{\perp}) \leq 12$. We can easily find such a bound by noting that if $l^2 = 2d$ then each of the 8 components l_i of l must have $l_i^2 \leq 2d$, so $-\sqrt{2d} \leq l_i \leq \sqrt{2d}$, and must be a half-integer: that gives

$$N_{E_8}(2d) \leq (2\lfloor 2\sqrt{2d} \rfloor + 1)^8 \tag{4}$$

For $d = 52$, this bound is about $8 \cdot 10^{12}$.

If we are a bit more precise, and note that the components of l are either all integers, or all proper (i.e. non-integer) half-integers, we save a factor 2^7, giving a bound of about $5 \cdot 10^{10}$. This is within reach of a brute-force search, but it is still high, especially considering that we have to do some substantial work for each candidate (compute the inner product with 240 different vectors[1]).

Thus an exhaustive search of all vectors in E_8 of length ≤ 60 is not computationally impossible but it would be cumbersome and would not extend to even slightly larger problems such as other cases of Question 1. The Weyl group $W(E_8)$ has order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600$ and should be used to reduce the size of the problem. There are two approaches to doing this.

(A) Randomised search. This is what was actually done in [6] and [8]. Since the non-existence of a vector l gives no information about the moduli space, we are willing to accept a very small probability of failing to detect such a vector. We therefore choose a large number of vectors of length less than $2 \cdot 61$ at random and expect that, as the Weyl group orbits are large, every orbit will be represented.

This approach worked very fast, using only a laptop computer and immediately available software (Maple). A search of twenty thousand randomly chosen vectors found all the pairs $(d, \#R(l^\perp))$ in the ranges wanted within the first two thousand iterations, in approximately two minutes. That is fairly convincing practical evidence that there are no more. Unfortunately the output was then mistranscribed, leading to the omission of the case $d = 52$ and the erroneous (but not really misleading) statement in [6] that “an extensive computer search for vectors orthogonal to at least 2 and at most 14 roots for other d has not found any”.

It is noteworthy that a similar search in the case $\Lambda = E_7$ did find some cases not discovered analytically, and for which a constructive method of finding l is still not known. In other words, some cases of the main theorem of [8] still have only a computer proof, although once l has been found it is easy enough to verify its properties by hand.

It is not so easy to estimate the probability *a priori* that a Weyl orbit might be missed. The Weyl group of $R(l^\perp)$, which is a subgroup of the Weyl group of E_8, obviously stabilises l and has order no more than 24 if $\#R(l^\perp) \leq 12$, but in principle the stabiliser of l in $W(E_8)$ could be much larger. In that case the Weyl group orbit would be small and more easily missed. In practice the randomised method seems to find all the orbits.

(B) Exhaustive search. The first author organised an exhaustive search, exploiting the Weyl group by searching a fundamental domain for the subgroup $H < W(E_8)$ generated by permutations of the eight components l_i and sign changes of an even number of components. This subgroup H has size $2^7 \cdot 8!$, so index 135 in $W(E_8)$: it gives us most of the symmetries, with very little effort.

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[1] We can be a lot more efficient than that, and skip most of these inner products, but even then we still have to compute dozens of inner products per candidate vector.

We say that $l \in E_8$ is in *normal form* if its components are all nonnegative (except possibly the first, l_1) and the squares of the components are nondecreasing from low index to high index. By acting with an element of H, we can translate any $l \in E_8$ to one in normal form: first permute the components, so their squares are in order; then make them all (but l_1) nonnegative, by changing the sign of every negative component (except l_1), and flipping the sign of l_1 once for every such change.

It is straightforward to enumerate the elements of length $2d$ in E_8 that are in normal form. For brevity, we will describe this only for the ones having integer components (one can get the ones with proper half-integer components in a very similar manner).

Step 1. For every index $i \neq 1$, in descending order, we consider all the possible values of l_i: we require l_i to be a non-negative integer such that

- its square, added to the sum of the squares of the coordinates that have been chosen (i.e. the l_j^2 with $j > i$), does not exceed $2d$ (otherwise $l^2 > 2d$, for any further choice of coordinates); and
- (unless $i = 8$) it is not greater than l_{i+1} (otherwise l would not be in normal form).

In other words, we let l_i take any value $s \in \mathbb{Z}$ such that

$$0 \leq s \leq \min\left\{ l_{i+1}, \sqrt{2d - \sum_{j>i} l_j^2} \right\}. \tag{5}$$

Step 2. See if $2d - \sum_{j=2}^{8} l_j^2$ is a perfect square m^2. If so, let l_1 take values $-m$ and m; if not, discard this choice of coordinates.

Step 3. Check whether the l so obtained are in E_8, i.e. whether $\sum_{j=1}^{8} l_j \in 2\mathbb{Z}$. Discard any that are not in E_8.

We must then filter these enumerated $l \in E_8$ to find the ones with $\#R(l^{\perp})$ in the required range ($2 \leq \#R(l^{\perp}) \leq 12$ for the case considered in [6]): this part of the procedure is exactly the same as for the randomised version. Since the roots come in pairs $\pm r$ it is enough to take inner products with a prepared list of positive roots (120 or them), and of course we can stop examining l as soon as we find a seventh pair of roots orthogonal to it.

The first author implemented this search in a high-level programming language (Haskell). Without spending much time optimising, this runs fast enough (a second or so on commercial hardware, for each of the low values of d we are interested in, namely $d \leq 60$). The partial use of the symmetries of E_8 is crucial, though: to go through all the vectors of given length $2d$ would have taken weeks or months for a single value of d.

This program discovered the lost case $d = 52$ and therefore Theorem 1. A variant of it for E_7 reconfirmed the results obtained by the randomised method in [8]. The code used for the E_8 case is given in Appendix B.

2.2. Further results

The exhaustive algorithm (B) from Section 2.1 can be modified to compute, in reasonable time, answers to some of the questions from Section 1.1 for small values of the parameters. We investigated Question 2 and Question 6 for small m and d with $\Lambda = E_7$ and $\Lambda = E_8$. For $\Lambda = E_8$ we also investigated Question 5 for the particular case $m = 14$, corresponding to canonical forms on $\mathcal{F}_{2d}$.

Specifically, we have so far computed the root type $P(\Lambda, 2d)$ for $\Lambda = E_7$ and $\Lambda = E_8$ and $d \leq 150$, and the first part of the root type (whether $m \in P(\Lambda, 2d)$ for $2 \leq m \leq 20$, say) for larger d, up to about 300 (further for some values of d). This part of the computation is fairly fast and only minor changes to the program are needed.

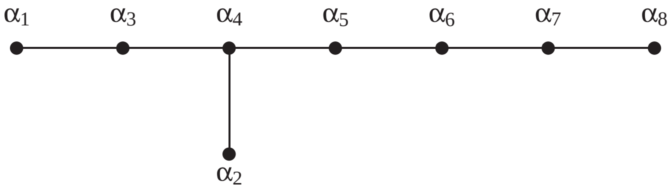
A little more work, and more computer time, is needed for Question 5. We must work now with $W(E_8)$, not with H, and we first compute a transversal for $W(E_8) : H$ (representatives for each of the 135 left cosets of H) and then reduce each of the 135 translates of each l to standard form before comparing them.

The outcome counts the number of ways of obtaining a canonical form on $\mathcal{F}_{2d}$ by quasi-pullback of Φ_{12}. There is no assurance either that the forms so obtained are linearly independent or that there are not more canonical forms that do not arise this way. The results are nevertheless intriguingly unpredictable. There are no such vectors for $d < 40$. There is such a vector for $d = 40$, and also for $d = 42,\ 43,\ 48$ (two orbits), $49,\ 51$–$54,\ 55$ and 56 (two orbits each), 57 and 59. There is no such vector for $d = 60$, but for 61 there are three orbits and thereafter the number of orbits drifts upwards irregularly. Without further comment, we tabulate below the number ν_{14} of $W(E_8)$ orbits of length $2d$ vectors in E_8 orthogonal to exactly 14 roots for $61 \leq d \leq 150$.

d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}
61	3	76	1	91	5	106	2	121	4	136	8
62	1	77	2	92	3	107	6	122	5	137	7
63	2	78	1	93	2	108	3	124	5	138	5
64	2	79	4	94	4	109	6	124	3	139	11
65	0	80	2	95	3	110	0	125	6	140	5
66	2	81	2	96	4	111	6	126	8	141	6
67	1	82	2	97	2	112	6	127	6	142	8
68	2	83	3	98	3	113	5	128	6	143	3
69	2	84	5	99	2	114	3	129	7	144	8
70	1	85	4	100	4	115	7	130	4	145	8
71	2	86	4	101	5	116	6	131	9	146	7
72	2	87	3	102	5	117	2	132	2	147	11
73	1	88	2	103	5	118	6	133	8	148	5
74	3	89	3	104	4	119	9	134	9	149	10
75	3	90	2	105	4	120	8	135	5	150	6

Appendix A. *d = 46, 50, 52, 54, 57,* by V. Gritsenko

In this appendix we find a vector $l \in E_8$ of square $2d$ orthogonal to exactly 12 roots in E_8, where d is as in the title of the appendix. (See [6] and [8] for the general context of this question.) We use below the combinatorics of the Dynkin diagram of E_8. We take the Coxeter basis of simple roots in E_8 as in [3]:



where $(e_1, \dots, e_8)$ is a Euclidean basis in the lattice $\mathbb{Z}^8$ and

$$\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7),$$
$$\alpha_2 = e_1 + e_2, \quad \alpha_k = e_{k-1} - e_{k-2} \ (3 \leq k \leq 8).$$

The lattice E_8 contains 240 roots. We recall that any root is a sum of simple roots with integral coefficients of the same sign. The fundamental weights ω_j of E_8 form the dual basis in $E_8 = E_8^\vee$, so $(\alpha_i, \omega_j) = \delta_{ij}$. The formulae for the weights are given in [3, Tabl. VII]. The Cartan matrix of the dual basis is

$$((\omega_i, \omega_j)) = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}. \tag{6}$$

We consider the two following cases when the orthogonal complement of a vector l in E_8 contains exactly 12 roots: $R(l^\perp_{E_8}) = A_2 \oplus 3A_1$ or $A_2 \oplus A_2$. (We note that $\#R(A_1) = 2$ and $\#R(A_2) = 6$.)

The cases *d = 46, 50, 54, 57.* There are four possible choices of the subsystem $A_2 \oplus 3A_1$ inside the Dynkin diagram of E_8 according to the choices of simple roots of A_2, namely $A_2^{(1,3)} = \langle \alpha_1, \alpha_3 \rangle$, $A_2^{(2,4)} = \langle \alpha_2, \alpha_4 \rangle$, $A_2^{(5,6)} = \langle \alpha_5, \alpha_6 \rangle$ or $A_2^{(7,8)} = \langle \alpha_7, \alpha_8 \rangle$. If A_2 is fixed then the three pairwise orthogonal copies of A_1 in the Dynkin diagram are defined automatically.

First, we consider $A_2^{(5,6)} = \langle \alpha_5, \alpha_6 \rangle$. Then $3A_1^{(5,6)} = \langle \alpha_2 \rangle \oplus \langle \alpha_3 \rangle \oplus \langle \alpha_8 \rangle$. Moreover $A_2^{(5,6)} \oplus 3A_1^{(5,6)}$ is the root system of the orthogonal complement of the vector $l_{5,6} = \omega_1 + \omega_4 + \omega_7 \in E_8$. In fact, if $r = \sum_{i=1}^{8} x_i \alpha_i$ is a positive root ($x_i \geq 0$) then $(r, l_{5,6}) = x_1 + x_4 + x_7 = 0$. Therefore $x_1 = x_4 = x_7 = 0$ and r belongs to $A_2^{(5,6)} \oplus 3A_1^{(5,6)}$.

Using the Cartan matrix (6) we obtain that $l_{5,6}^2 = 2 \cdot 46$. Doing similar calculations with the other three copies of A_2 given above we find

$$l_{1,3} = \omega_4 + \omega_6 + \omega_8, \quad l_{2,4} = \omega_3 + \omega_5 + \omega_7, \quad l_{7,8} = \omega_1 + \omega_4 + \omega_6$$

with $l_{1,3}^2 = 2 \cdot 50$, $l_{2,4}^2 = 2 \cdot 54$ and $l_{7,8}^2 = 2 \cdot 57$.

The case *d* = *52*. We consider the sublattice $M = A_2 \oplus A_2 = \langle \alpha_3, \alpha_4 \rangle \oplus \langle \alpha_6, \alpha_7 \rangle$ in E_8. Then M is the root system of the orthogonal complement of the vector $l_M = \omega_1 + \omega_2 + \omega_5 + \omega_8$ with $l_M^2 = 2 \cdot 52$.

Appendix B. The computer code

Below is the code used to check the combinatorial problem from [6], and thus to find Theorem 1. The programs were written in the functional programming language Haskell (`http://www.haskell.org`). The web page

`http://people.bath.ac.uk/masgks/Rootcounts`

contains links to further code and output.

```
{-# LANGUAGE TypeSynonymInstances,NoImplicitPrelude #-}
module E8 where

import qualified Algebra.Ring
import           Control.Applicative      ((<$>),(<*>))
import qualified Data.Vector          as V
import           Data.List                (intercalate,nubBy)
import qualified Data.MemoCombinators as Memo
import           Data.Ratio
  (Ratio,numerator,denominator,(%))
import qualified Data.Set             as Set
import           Data.Typeable            (Typeable)
import           Math.Combinatorics.Species
  (ksubsets,set,ofSize,enumerate,Set(getSet,Set),Prod(Prod))
import           MyPrelude hiding (numerator,denominator,(%))
import qualified Prelude
import           System.Environment       (getArgs)
import qualified Algebra.Additive

 -- Some types and helper functions for dealing with
 -- "vectors" (implemented as arrays of rational numbers).

type Coordinate
  = Ratio Int
```

```
type Vector
  = V.Vector Coordinate

-- Inner product.
inp :: Vector -> Vector -> Coordinate
inp a b = V.sum (V.zipWith (*) a b)

half :: Coordinate
half = 1 % 2

-- Product of scalar with vector.
l :: Coordinate -> Vector -> Vector
l = V.map . (*)

instance Algebra.Additive.C Vector where
  (+) = V.zipWith (+)
  (-) = V.zipWith (-)
  negate = l (-1)
  zero = V.fromList [0,0,0,0,0,0,0,0]

-- Some data regarding E_8

delta :: (Eq a,Algebra.Ring.C b) => a -> a -> b
delta i j = if i == j then 1 else 0

-- 'e i' gives the i'th standard basis vector of R_8.
e :: Int -> Vector
e i = V.fromList $ map (delta i) [1 .. 8]

-- This is the usual integral basis of the lattice E_8.
basis :: [Vector]
basis =
  [
    l half $ (e 1 + e 8) - (sum $ map e [2 .. 7])
  , e 1 + e 2
  ] ++ map (\ i -> e (i - 1) - e (i - 2)) [3 .. 8]

roots :: [Vector]
roots = d8 ++ x118 where
  d8 = concatMap ((\ [a,b] ->
    [a + b,a - b,b - a,negate a - b]) . map e . getSet) $
      enumerate (ksubsets 2) [1 .. 8]
  x118 = map (\ (Prod (Set neg) (Set pos)) ->
    l half $ sum (map (negate . e) neg) + sum (map e pos)) $
      enumerate ((set `ofSize` even) * set) [1 .. 8]
```

```
-- 'posRoots' contains exactly one of every pair
-- (a,-a) of roots.
posRoots :: [Vector]
posRoots = nubBy (\ a b -> a == b || a == negate b) roots

-- Generate elements l of the E_8 lattice with the property
-- that l^2 = 2 d.  We need only one element of each orbit
-- under the action of the Weyl group.  In particular, we
-- may assume that all coordinates but one (say, the first)
-- are nonnegative, and that the successive coordinates are
-- nondecreasing.  We generate exactly one element of each
-- H-orbit, where H is the subgroup of permutations and even
-- sign changes.

gen :: Int -> [Vector]
gen d = genInt d ++ genHalfInt d

genInt :: Int -> [Vector]
genInt d = map (V.fromList . map fromIntegral) $ go [] 0 where
  -- Given the length of a partial vector, compute the maximal
  -- new coordinate which does not increase the length of the
  -- vector beyond 2 d.
  maxCoord :: Int -> Int
  maxCoord s = floor (sqrt (fromIntegral $ dD - s) :: Double)

  dD :: Int
  dD = 2 * d

  -- We maintain a list of coordinates chosen so far, every
  -- one together with the sum of squares of the coordinates
  -- up to and including that coordinate.
  -- The generated vectors are elements of E_8, because the
  -- sum of the squares of their components is even, hence
  -- the sum of the components as well.
  go :: [(Int,Int)] -> Int -> [[Int]]
  -- We have fixed all eight coordinates.
  go fixed@((_,sq) : ps) 8
    -- The vector has the right length; add the relevant
    -- solutions (using 'vary'), and continue searching.
    | sq == dD  = vary (map fst fixed) ++ lower ps 7
    -- The vector has the wrong length, continue searching.
    | otherwise = lower ps 7
  go fixed               n = let
    (m,s) = case fixed of
```

```
      []        -> (maxCoord 0,0)
      (c,s) : _ -> (Prelude.min (maxCoord s) c,s)
   in
    go ((m,s + m ^ 2) : fixed) (n + 1)

  -- Lexicographically decrease the given vector, and continue
  -- the generation from there.
  lower :: [(Int,Int)] -> Int -> [[Int]]
  lower []           _ = []
  lower ((x,s) : ps) n
    | x == 0    = lower ps (n - 1)
    | otherwise = go ((x - 1,s + 1 - 2 * x) : ps) n

  vary :: [Int] -> [[Int]]
  vary (x : xs) = if x == 0
    then [0 : xs]
    else [x : xs,negate x : xs]

-- For vectors with all coordinates half-integers, we work
-- with the doubles of the coordinates.
genHalfInt :: Int -> [Vector]
genHalfInt d = map (V.fromList . map (% 2)) $ go [] 0 where
  maxCoord :: Int -> Int
  maxCoord = Memo.integral m where
    m s = f $ floor (sqrt (fromIntegral $ dE - s) :: Double)
    f k = if odd k then k else k - 1

  dE :: Int
  dE = 8 * d

  go :: [(Int,Int)] -> Int -> [[Int]]
  go fixed@((_,sq) : ps) 8
    | sq == dE  = filter e8 (vary $ map fst fixed)
               ++ lower ps 7
    | otherwise = lower ps 7
  go fixed                   n = let
    (m,s) = case fixed of
      []        -> (maxCoord 0,0)
      (c,s) : _ -> (Prelude.min (maxCoord s) c,s)
   in
    go ((m,s + m ^ 2) : fixed) (n + 1)

  -- Decides whether a given vector is an element of E_8
  e8 :: [Int] -> Bool
  e8 = (== 0) . flip rem 4 . sum
```

```
lower :: [(Int,Int)] -> Int -> [[Int]]
lower []            _ = []
lower ((x,s) : ps) n
  | x == 1    = lower ps (n - 1)
  | otherwise = go ((x - 2,s + 4 - 4 * x) : ps) n

vary :: [Int] -> [[Int]]
vary (x : xs) = [x : xs,negate x : xs]
```

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