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N. Ujević

AN APPLICATION OF THE MONTGOMERY IDENTITY TO QUADRATURE RULES

Abstract. We use the Montgomery identity to obtain an optimal quadrature rule. It turns out that this rule is the well-known compound trapezoidal rule.

1. Introduction

The Montgomery identity is recently considered by many authors ([1]-[5]). This identity has the form

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b K(x,t)f'(t)dt,$$

where

$$K(t)=\begin{cases} t-a, & t\in[a,x] \\ t-b, & t\in(x,b] \end{cases}.$$

From this identity we can obtain the well known integral Ostrowski inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\|f'\|_\infty(b-a).$$

This inequality is considered in really many recently published papers. It gives an error bound for a simple quadrature rule.

In this paper we use the Montgomery identity and derive an optimal quadrature rule with respect to a given way of estimation of its error. It turns out that this optimal quadrature rule is in fact the well-known compound trapezoid rule. In such a way we proved that the compound trapezoidal rule is the best possible in a given class of quadrature rules if we a priory give the way of estimation of remainder terms for these rules. In this sense the results of this paper can be considered as a generalization of results obtained in [6].

2. Quadrature rules

First we consider a problem in the interval $[0,1]$. Let $\{0=x_0<x_1<\cdots<x_n=1\}$ be a partition of $[0,1]$ and

$$\sum_{i=0}^{n} w_i=1,\ w_i\in R. \tag{1}$$

If $x_i \in [0,1]$ then, using the Montgomery identity, we have

$$f(x_i) = \int_0^1 f(t)dt + \int_0^1 K(x_i,t)f'(t)dt, \tag{2}$$

where

$$K_0(t) = t-1, t \in [0,1],$$

$$K_i(t) = \begin{cases} t, & t \in [0,x_i] \\ t-1, & t \in (x_i,1] \end{cases},$$

for $i = 1,2,...,n-1$, $x_i \in [0,1]$ and

$$K_n(t) = t, t \in [0,1].$$

From (2) we get

$$\sum_{i=0}^{n} w_i f(x_i) = \int_0^1 f(t)dt + \int_0^1 K(t)f'(t)dt,$$

where

$$K(t) = \sum_{i=0}^{n} w_i K(x_i,t). \tag{3}$$

The function $K(t)$ can be written in the equivalent form

$$K(t) = \begin{cases} t - w_0, & t \in [0,x_1] \\ t - \sum_{j=0}^{1} w_j, & t \in (x_1,x_2] \\ t - \sum_{j=0}^{2} w_j, & t \in (x_2,x_3] \\ \cdots & \\ t - \sum_{j=0}^{n-1} w_j, & t \in (x_{n-1},1] \end{cases}.$$

We have

$$\left| \int_0^1 f(t)dt - \sum_{i=0}^{n} w_i f(x_i) \right| \le \|f'\|_\infty \int_0^1 |K(t)|\, dt.$$

We now consider the minimizing problem

$$\int_0^1 |K(t)|\, dt \to \min. \tag{4}$$

If we introduce the notation

$$\sigma_i = \sum_{j=0}^{i} w_j$$

then the problem (4) can be written in the form

$$\int_0^{x_1} |t-\sigma_0|\,dt+\cdots+\int_{x_{n-1}}^1 |t-\sigma_{n-1}|\,dt \to \min. \tag{5}$$

We required that $\sigma_i \in [x_i, x_{i+1}]$, $i=0,1,...,n-1$. Then we have

$$\begin{aligned} \int_{x_i}^{x_{i+1}} |t-\sigma_i|\,dt &= \int_{x_i}^{\sigma_i} (\sigma_i - t)\,dt + \int_{\sigma_i}^{x_{i+1}} (t-\sigma_i)dt \\ &= \frac{1}{2}\left[(\sigma_i - x_i)^2 + (x_{i+1}-\sigma_i)^2\right]. \end{aligned} \tag{6}$$

Using the relation (6) we can write the problem (5) in the form

$$F(w_0,...,w_{n-1}) = \frac{1}{2}\sum_{i=0}^{n-1}\left[(\sigma_i - x_i)^2 + (x_{i+1}-\sigma_i)^2\right] \to \min.$$

We now solve this problem. For that purpose we calculate

$$\frac{\partial F}{\partial w_j} = \sum_{i=0}^{n-1}\left[(\sigma_i - x_i)\frac{\partial \sigma_i}{\partial w_j} - (x_{i+1}-\sigma_i)\frac{\partial \sigma_i}{\partial w_j}\right], \tag{7}$$

for $j=0,1,2,...,n-1$. Since

$$\frac{\partial \sigma_i}{\partial w_j} = \begin{cases} 1, & j \le i \\ 0, & j > i \end{cases}$$

from (7) we get

$$\begin{aligned} \frac{\partial F}{\partial w_0} &= \sum_{i=0}^{n-1}[2\sigma_i - (x_i + x_{i+1})] \\ \frac{\partial F}{\partial w_1} &= \sum_{i=1}^{n-1}[2\sigma_i - (x_i + x_{i+1})] \\ &\cdots \\ \frac{\partial F}{\partial w_{n-1}} &= 2\sigma_{n-1} - (x_{n-1} + x_n). \end{aligned}$$

If we wish to find the minimum we have to require that

$$\frac{\partial F}{\partial w_j} = 0,\ j=0,1,2,...,n-1.$$

We also introduce the notations

$$\begin{aligned} p_j &= \sum_{i=j}^{n-1}(x_i + x_{i+1}), \\ q_j &= \sum_{i=j}^{n-1}\sigma_i, \end{aligned}$$

for $j = 0, 1, 2, ..., n-1$. The we can write the above system in the form

$$\begin{aligned} 2q_0 - p_0 &= 0 \\ 2q_1 - p_1 &= 0 \\ &\cdots \\ 2q_{n-1} - p_{n-1} &= 0. \end{aligned}$$

Since

$$\begin{aligned} p_j - p_{j+1} &= x_j + x_{j+1}, \\ q_j - q_{j+1} &= \sigma_j, \end{aligned}$$

we have

$$\begin{aligned} 2\sigma_0 - (x_0 + x_1) &= 0 \\ 2\sigma_1 - (x_1 + x_2) &= 0 \\ &\cdots \\ 2\sigma_{n-1} - (x_{n-1} + x_n) &= 0. \end{aligned}$$

It is not difficult to solve the above system. We have

$$\sigma_0 = w_0 = \frac{x_0 + x_1}{2},$$

$$\sigma_1 = w_0 + w_1 = \frac{x_1 + x_2}{2}$$

such that

$$w_1 = \frac{x_2 - x_0}{2}.$$

Generally, we have

$$w_j = \frac{x_{j+1} - x_{j-1}}{2},\ j = 2, 3, .., n-1,$$

and

$$w_n = \frac{1 - x_{n-1}}{2},$$

since (1) holds.

In fact, we get the next formula

$$\int_0^1 f(t)dt = \sum_{i=0}^{n} w_i f(x_i) - \int_0^1 K(t) f'(t)dt,$$

where $K(t)$ is given by (3) and w_i are given above. Hence, we got the following result.

THEOREM 1. *Let* $f \in C^1(0,1)$. *Then*

$$\int_0^1 f(t)dt = \frac{h_0 f(0) + h_{n-1} f(1)}{2} + \sum_{i=1}^{n-1} \frac{h_{i-1} + h_i}{2} f(x_i) + R(f), \tag{8}$$

where

$$R(f) = -\int_0^1 K(t)f'(t)dt,$$

$K(t)$ *is given by (3),* $h_i = x_{i+1} - x_i$, $i = 0,1,2,...,n-1$ *and*

$$|R(f)| \le \frac{\|f'\|_\infty}{4}\sum_{i=0}^{n-1} h_i^2. \tag{9}$$

Proof. The relation (8) is proved above. We have

$$\begin{aligned}\int_0^1 |K(t)|\,dt &= \sum_{i=0}^{n-1}\left[\int_{x_i}^{\sigma_i}(\sigma_i - t)\,dt + \int_{\sigma_i}^{x_{i+1}}(t-\sigma_i)dt\right]\\ &= \frac{1}{2}\sum_{i=0}^{n-1}\left[(\sigma_i - x_i)^2 + (x_{i+1}-\sigma_i)^2\right]\\ &= \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{4},\end{aligned}$$

since

$$\sigma_i = \frac{x_i + x_{i+1}}{2},\ i = 0,1,2,...,n-1.$$

Hence, the relation (9) holds, too. ☐

COROLLARY 1. *Let* $f \in C^1(0,1)$. *Then*

$$\int_0^1 f(t)dt = \frac{f(0)+f(1)}{2}h + h\sum_{i=1}^{n-1} f(x_i) + R_1(f), \tag{10}$$

where

$$R_1(f) = -\int_0^1 K(t)f'(t)dt,$$

$K(t)$ *is given by (3),* $h = x_{i+1} - x_i = 1/n$, $i = 0,1,2,...,n-1$ *and*

$$|R_1(f)| \le \frac{h^2}{4}\|f'\|_\infty = \frac{\|f'\|_\infty}{4n}.$$

REMARK 1. Note that the quadrature rule given in Corollary 1 is the well-known compound trapezoidal quadrature rule. From the above considerations we can also conclude that this rule is optimal in the sense that it has a minimal error bound when we estimate this error in the described way.

In fact, in both above cases we can obtain a better estimation of the error.

THEOREM 2. *Let the assumptions of Theorem 1 and Corollary 1 hold. If* $\gamma,\Gamma \in$ *R are numbers such that* $\gamma \le f'(t) \le \Gamma$, $t \in [0,1]$ *then*

$$|R(f)| \le \frac{\Gamma-\gamma}{8}\sum_{i=0}^{n-1} h_i^2, \tag{11}$$

$$|R_1(f)| \leq \frac{h^2}{8}(\Gamma - \gamma) \tag{12}$$

Proof. We have

$$\begin{aligned}\int_0^1 K(t)dt &= \int_0^{x_1}(t-\sigma_0)\,dt + \cdots + \int_{x_{n-1}}^1 (t-\sigma_{n-1})\,dt \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\left[(\sigma_i - x_{i+1})^2 - (x_i - \sigma_i)^2\right] \\ &= 0,\end{aligned}$$

since

$$\sigma_i = \frac{x_i + x_{i+1}}{2},\ i = 0,1,2,...,n-1.$$

Thus,

$$\int_0^1 K(t)(f'(t) - \frac{\Gamma+\gamma}{2})dt = \int_0^1 K(t)f'(t)$$

and

$$\int_0^1 \left|K(t)(f'(t) - \frac{\Gamma+\gamma}{2})\right| \leq \frac{\Gamma-\gamma}{2}\int_0^1 |K(t)|\,dt,$$

since

$$\left\|f' - \frac{\Gamma+\gamma}{2}\right\|_\infty \leq \frac{\Gamma-\gamma}{2}.$$

Now it is not difficult to see that (11) and (12) hold. ☐

If we write the above rule in the interval $[a,b]$ then we get the next result.

THEOREM 3. *Let $f \in C^1(a,b)$. Then*

$$\int_a^b f(t)dt = \frac{h_0 f(a) + h_{n-1} f(b)}{2} + \sum_{i=1}^{n-1}\frac{h_{i-1}+h_i}{2}f(x_i) + R(f),$$

where $h_i = x_{i+1} - x_i$, $i = 0,1,2,...,n-1$ and

$$|R(f)| \leq \frac{\Gamma-\gamma}{8}\sum_{i=0}^{n-1}h_i^2,$$

for $\gamma, \Gamma \in R$ such that $\gamma \leq f'(t) \leq \Gamma$, $t \in [a,b]$.

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Nenad UJEVIĆ, Department of Mathematics, University of Split, Teslina 12/III, 21000 Split, CROATIA
e-mail: `ujevic@pmfst.hr`

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