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WEIGHTED POWER MEANS OF OPERATORS

Abstract. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be positive real numbers satisfying $\sum_{i=1}^{n} \sigma_i = 1$ and $A_1, A_2, \cdots, A_n$ be positive operators. Let $M_{\sigma,\gamma}(\underline{A}) = (\sum_{i=1}^{n} \sigma_i A_i^{\gamma})^{1/\gamma}$, $\gamma > 0$. It has been shown that $\lim_{\gamma \to 0+} M_{\sigma,\gamma}$ exists. Some known inequalities have also been generalized.

1. Introduction

K. V. Bhagwat and R. Subramanian [2] considered the validity of the well known inequalities between power means of a set of positive real numbers, when the latter are replaced by positive operators on a Hilbert space. They proved an analogue of the arithmetic-harmonic mean inequality for positive numbers in the case of positive operators. In what follows, we extend the above said inequality to weighted means. The procedure for extension of the arithmetic-harmonic mean inequality to weighted means suggested in [2], though standard is yet involved. It turns out that the method of proof employed in [2] when suitably modified, gives a direct proof of weighted arithmetic-harmonic mean inequality.

We show that $\lim_{\gamma \to 0+} M_{\sigma,\gamma}(\underline{A})$, where $M_{\sigma,\gamma}(\underline{A}) = (\sum_{i=1}^{n} \sigma_i A_i^{\gamma})^{1/\gamma}$, σ_i $(i = 1, 2, ..., n)$ are non-negative numbers with $\sum_{i=1}^{n} \sigma_i = 1$ and A_i $(i = 1, 2, ..., n)$ are positive operators, exists and equals $M_{\sigma,0}(\underline{A}) = \exp\{\sum_{i=1}^{n} \sigma_i \log A_i\}$. The existence of this limit has been proved in [4], however, the method of proof therein is rather long and tedious. We give a direct proof of the existence of the limit. The value of the limit reduces to the usual generalized geometric mean in the case of commuting operators. The surprising fact in the case of Hilbert space operators is that the generalized geometric mean is not less than or equal to the weighted arithmetic mean. There is a reference to generalized geometric mean in [5] as well.

Some other inequalities proved in [2] have also been extended to weighted means.

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2. Weighted Power Means

Throughout $\mathcal{H}$ will denote a Hilbert space and $A_j,\ 1 \le j \le n$, are bounded self adjoint positive definite linear operators from $\mathcal{H}$ to $\mathcal{H}$. Let $\sigma_j,\ 1 \le j \le n$, be positive real numbers such that $\sum\limits_{j=1}^{n} \sigma_j = 1$ and $\gamma \neq 0$ be a real number. Set $M_{\sigma,\gamma}(\underline{A}) = (\sum\limits_{j=1}^{n} \sigma_j A_j^{\gamma})^{1/\gamma}$. The following theorem holds.

THEOREM 1. *With notations as in paragraph above, we have*

$$\lim_{\gamma\to 0+} (\sum_{j=1}^{n} \sigma_j A_j^{\gamma})^{1/\gamma} = \exp\{\sum_{j=1}^{n} \sigma_j \log A_j\}.$$

Proof. Observe that

$$\begin{aligned}
\sum_{j=1}^{n} \sigma_j A_j^{\gamma} &= \sum_{j=1}^{n} \sigma_j \exp\{\gamma \log A_j\} \\
&= I + \gamma \sum_{j=1}^{n} \sigma_j \log A_j + \sum_{j=1}^{n} \sigma_j \sum_{k=2}^{\infty} \frac{(\gamma \log A_j)^k}{k!} \\
&= I + \gamma \sum_{j=1}^{n} \sigma_j \log A_j + \sum_{k=2}^{\infty} \frac{\gamma^k}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k .
\end{aligned}$$

So,

$$\begin{aligned}
&\left(\sum_{j=1}^{n} \sigma_j A_j^{\gamma}\right)^{1/\gamma} = \left(I + \gamma\{\sum_{j=1}^{n} \sigma_j \log A_j + \sum_{k=2}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k\}\right)^{1/\gamma} \\
&= \exp\left\{ \frac{1}{\gamma} \log(I + \gamma\{\sum_{j=1}^{n} \sigma_j \log A_j + \sum_{k=2}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k\}) \right\} \\
&= \exp\left\{ \frac{1}{\gamma} (\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \gamma^l \{\sum_{j=1}^{n} \sigma_j \log A_j + \sum_{k=2}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k\}^l) \right\} \\
&= \exp\left\{ \sum_{j=1}^{n} \sigma_j \log A_j + \sum_{k=2}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k \right. \\
&\qquad \left. + \gamma \sum_{l=2}^{\infty} \frac{(-1)^{l-1}}{l} \gamma^{l-2} \{\sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log A_j)^k\}^l \right\}.
\end{aligned}$$

Notice that for $0 < \gamma < 1$,

$$\begin{aligned}
||\sum_{k=2}^{\infty} \frac{\gamma^{k-1}}{k!} \sum_{j=1}^{n} \sigma_j (\log \mathrm{A}_j)^k|| &\le \gamma \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{j=1}^{n} \sigma_j ||\log A_j||^k \\
&\le \gamma \sum_{k=2}^{\infty} \frac{1}{k!} \max_{1\le j\le n} ||\log A_j||^k \\
&\le \gamma \left(\exp\{\max_{1\le j\le n} ||\log A_j||\} - \min_{1\le j\le n} ||\log A_j|| - 1 \right)
\end{aligned}$$

and

$$||\gamma\sum_{l=2}^{\infty}\frac{(-1)^{l-1}}{l}\gamma^{l-2}\left(\sum_{k=1}^{\infty}\frac{\gamma^{k-1}}{k!}\sum_{j=1}^{n}\sigma_j(\log A_j)^k\right)^l ||$$
$$\leq \sum_{l=2}^{\infty}||\sum_{k=1}^{\infty}\frac{\gamma^k}{k!}\sum_{j=1}^{n}\sigma_j(\log A_j)^k||^l$$
$$\leq \sum_{l=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\gamma^k}{k!}\max_{1\leq j\leq n}||(\log A_j)^k||\right)^l$$
$$\leq \sum_{l=2}^{\infty}\left(\exp(\gamma\max_{1\leq j\leq n}||(\log A_j||-1)\right)^l$$
$$\leq \sum_{l=0}^{\infty}\left(\exp(\gamma\max_{1\leq j\leq n}||(\log A_j||-1)\right)^l$$
$$-1-\{\exp(\gamma\max_{1\leq j\leq n}||(\log A_j||)-1\}$$
$$=\left\{\frac{1}{1-\left(\exp\left(\gamma\max_{1\leq j\leq n}||\log A_j||\right)-1\right)}\right\}-\exp(\gamma\max_{1\leq j\leq n}||\log A_j||.$$

Consequently, $\left(\sum_{j=1}^{n}\sigma_j A_j^{\gamma}\right)^{1/\gamma}$ tends to $\exp\{\sum_{j=1}^{n}\sigma_j\log A_j\}$ as $\gamma\to 0+$. □

REMARK 1. The generalized geometric mean, namely, $\exp\{\sum_{j=1}^{n}\sigma_j\log A_j\}$, is not necessarily less than the weighted arithmetic mean. Indeed,

$$\exp\{\sum_{j=1}^{n}\sigma_j\log A_j\}\leq\sum_{j=1}^{n}\sigma_j A_j$$

implies the map $x\to e^x$ is operator convex, which as is well known, is false in general (see [3] Problem V.5.1, p.147).

We shall need the following lemma in the sequel. For a proof (see [3] p.114).

LEMMA 1. *If B is strictly positive and $A \geq B$ then A is strictly positive and* $B^{-1} \geq A^{-1} > 0$.

THEOREM 2. *For a set of positive operators $\{A_1, A_2, \cdots, A_n\}$ on a Hilbert space $\mathcal{H}$, a set of positive real numbers $\sigma_1, \sigma_2, \cdots, \sigma_n$ satisfying $\sum_{i=1}^{n} \sigma_i = 1$ and $q \geq p \geq 1$, $M_{\sigma,q}(\underline{A}) \geq M_{\sigma,p}(\underline{A})$, where $\underline{A} = (A_1, A_2, \cdots, A_n)$.*

Proof. The case $1 \leq p = q$ is trivial, so we prove only the result for $p < q$, i.e., $pq^{-1} < 1$. Set $A_j^q = B_j$ and $\alpha = pq^{-1}$.

$$(\sum_{j=1}^{n} \sigma_j A_j^q)^{\alpha} = (\sum_{j=1}^{n} \sigma_j B_j)^{\alpha} \geq \sum_{j=1}^{n} \sigma_j B_j^{\alpha} = \sum_{j=1}^{n} \sigma_j A_j^p\,,$$

using the concavity of the map $A \to A^{\alpha},\ 0 < \alpha < 1$, ([3] Th.V.2.5 & V.2.10). If $p \geq 1$, then the map $A \to A^{1/p}$ is order preserving (see [3]) and since $(\sum_{j=1}^{n} \sigma_j A_j^q)^{pq^{-1}} \geq \sum_{j=1}^{n} \sigma_j A_j^p$, it follows that

$$(\sum_{j=1}^{n} \sigma_j A_j^q)^{1/q} \geq (\sum_{j=1}^{n} \sigma_j A_j^p)^{1/p}.$$

□

COROLLARY 1. *For positive operators $A_i,\ i = 1, 2, ...n,\ q \leq p \leq -1$ and $\sigma_i > 0,\ i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \sigma_i = 1$, we have*

$$M_{\sigma,q}(\underline{A}) \leq M_{\sigma,p}(\underline{A})$$

where $\underline{A} = (A_1, A_2, ..., A_n)$.

Proof. Replacing A_j by A_j^{-1} in Theorem 2, we obtain

$$(\sum_{j=1}^{n} \sigma_j A_j^{-q_1})^{1/q_1} \geq (\sum_{j=1}^{n} \sigma_j A_j^{-p_1})^{1/p_1},$$

for $q_1 \geq p_1 \geq 1$. Since the right hand side of the above inequality is positive, using Lemma 1 and then replace $-q_1,\ -p_1$ by $q,\ p$ respectively, we obtain the desired inequality. □

THEOREM 3. *For positive operators $A_i,\ i = 1, 2, ..., n$ and positive real numbers $\sigma_i,\ i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \sigma_i = 1$, the following inequality holds*

$$M_{\sigma,\gamma}(\underline{A}) \leq M_{\sigma,1}(\underline{A}) \leq M_{\sigma,2\gamma}(\underline{A})$$

where $\underline{A} = (A_1, A_2, ..., A_n)$ *and* $1/2 \le \gamma \le 1$.

Proof. We need only to prove the first inequality as the last inequality is trivial from Theorem 2. Now, since $1/2 \le \gamma \le 1$, i.e., $1 \le \gamma^{-1} \le 2$ and the function $x \to x^{1/\gamma}$ is operator convex (see [3] Ex.V.2.11) it follows that

$$(\sum_{i=1}^{n} \sigma_i A_i^{\gamma})^{1/\gamma} \le \sum_{i=1}^{n} \sigma_i A_i,$$

i.e.,

$$M_{\sigma,\gamma}(\underline{A}) \le M_{\sigma,1}(\underline{A}).$$

□

COROLLARY 2. *For positive operators* A_i, $i = 1, 2, ..., n$, *and positive real numbers* $\sigma_i > 0$, $i = 1, 2, ..., n$ *with* $\sum_{i=1}^{n} \sigma_i = 1$, *we have*

$$M_{\sigma,2\gamma}(\underline{A}) \le M_{\sigma,-1}(\underline{A}) \le M_{\sigma,\gamma}(\underline{A})$$

for $-1 \le \gamma \le -1/2$.

Proof. From Theorem 3, we have

$$(\sum_{i=1}^{n} \sigma_i A_i^{\gamma_1})^{1/\gamma_1} \le \sum_{i=1}^{n} \sigma_i A_i \le (\sum_{i=1}^{n} \sigma_i A_i^{2\gamma_1})^{1/2\gamma_1}, \tag{1}$$

for $1/2 \le \gamma_1 \le 1$.

Replacing A_i by A_i^{-1} in (1), using Lemma 1 and then replacing γ_1 by $-\gamma$, we obtain the desired result. □

THEOREM 4. *For positive operators* A_i, $i = 1, 2, \cdots, n$ *and positive real numbers.* σ_i, $i = 1, 2, \cdots, n$ *with* $\sum_{i=1}^{n} \sigma_i = 1$ *the following inequality holds*

$$\left(\sum_{i=1}^{n} \sigma_i A_i^{-\gamma}\right)^{-1/\gamma} \le \left(\sum_{i=1}^{n} \sigma_i A_i^{\gamma}\right)^{1/\gamma},$$

for $\gamma \ge 1$.

Proof. Since the map $x \to x^{-1}$ is operator convex, it follows that

$$\left(\sum_{i=1}^{n} \sigma_i A_i^{-1}\right)^{-1} \le \sum_{i=1}^{n} \sigma_i A_i.$$

Now replacing A_i by A_i^{γ} and using the order perserving property of the map $x \to x^{\alpha}$, $0 < \alpha \le 1$, we obtain

$$(\sum_{i=1}^{n} \sigma_i A_i^{-\gamma})^{-1/\gamma} \le (\sum_{i=1}^{n} \sigma_i A_i^{\gamma})^{1/\gamma}.$$

□

Finally, we have the following theorem (c.f. [1]).

THEOREM 5. *Let $q,\ p$ be real numbers. such that one of the following holds*

(a) $q \geq p$ $\qquad$ $q \notin (-1,1),\ p \notin (-1,1)$;
(b) $q \geq 1 \geq p \geq 1/2$; $\qquad$ or
(c) $p \leq -1 \leq q \leq -1/2$.

Then

$$M_{\sigma,q}(\underline{A}) \geq M_{\sigma,p}(\underline{A}).$$

Proof. (a). If $q \geq p \geq 1$, the result follows from Theorem 2. If $-1 \geq q \geq p$, the result follows from corollary 1. In case $1 \leq -p \leq q$, we obtain from Theorem 2, Theorem 4 and Corollary 1 above the desired inequality. The case when $1 \leq q \leq -p$ may be dealt with similarly.
(b). On combining the results of Theorem 2 and Theorem 3 we obtain the desired inequality.
(c). On combining the results of Corollary 1 and Corollary 2 we obtain the desired inequality. □

REMARK 2. Negative weighted power means of positive operators can be defined by a suitable limiting process as described in [2]. Since the details are no different, they are hence not provided. All the inequalities established above hold for positive operators.

References

[1] ALIČ M., MOND B., PEČARIĆ J. AND VOLENEC V., *Bounds for the differences of matrix means*, SIAM J. Matrix Anal. Appl.**18** (1) (1997), 119–123.

[2] BHAGWAT K.V. AND SUBRAMANIAN R., *Inequalities between means of positive operators*, Proc. Camb. Phil. Soc., (1978), 393–401.

[3] BHATIA R., *Matrix analysis*, Springer-Verlag, 1997.

[4] NUSSBAUM R.D. AND COHEN J.E., *The arithmetic geometric mean and its generalizations for non-commuting linear operators*, Ann. Sci. Norm. Sup. Sci. IV **15** (2) (1989), 239–308.

[5] TRAPP G.E., *Hermition semidefinite matrix means and related inequalities*, Linear and Multilinear Alg. **16** (1984), 113–123.

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