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Preface

Two important research fields in Algebraic Geometry are the study of linear systems of curves with prescribed base points and the classification of defective varieties. These topics go back to the work of classical Italian authors, as F. Enriques, G. Scorza, B. Segre, C. Segre, F. Severi, A. Terracini, E. Togliatti and many others. These two research fields can be viewed in the more general setting of polynomial interpolation (the first one) and of projective embeddings (the second one).

On September 15–20, 2003 the meeting "School (and Workshop) on Polynomial Interpolation and Projective Embeddings" was held at theDepartment of Mathematics of the Politecnico diTorino.The School was articulated in two series lectures delivered by L. Chiantini and R. Miranda. C. Bocci also gave exercises sessions. During the Workshop many researchers gave contributions concerning their results in the field. This special issue contains research papers regarding the topics of the Workshop and abstracts of some of the short conferences given by the participants. The organizers would like to thank all thepartecipants to the School/ Workshop and the Departementof Mathematics for the warm hospitality. Special thanks go to themain speakers for their work before, during and after the School/ Workshop.The School/Workshop was partially supported by the national group GNSAGA of INdAM, by the Department of Mathematics of the Universities of Torino and Ferrara in the frameworkof the national project "Geometry on algebraic varieties" cofinanced by Italian MIUR and by EAGER.

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ON THE REPRESENTATION OF ENRIQUES SURFACES AS DOUBLE PLANES

Abstract. In this paper we give a short proof of the well-known representation of Enriques surfaces as double planes, by using the properties of the adjoint linear system to the branch curve.

Enriques surfaces play a fundamental role in the classification of complex algebraic surfaces: historically they have been the first examples of irrational surfaces with geometric genus $p_g = 0$ and irregularity $q = 0$. Indeed, in 1894, Enriques suggested in a letter to Castelnuovo that these properties were fulfilled by (the normalization of) a sextic surface in $\mathbb{P}^3(\mathbb{C})$ having the six edges of a tetrahedron as double lines. Soon later, in 1896, Castelnuovo proved his celebrated rationality criterion, which states that an algebraic surface is rational if and only if it is regular and has bi-genus $P_2 = 0$.

In 1906, Enriques proved in [10] that every surface with $P_2 = 1$ and $P_3 = q = 0$ is isomorphic to his original example and he gave a rather complete treatment of these surfaces, which have justly been named after him. In particular Enriques showed that they can be represented as *double planes*, i.e. as double covers of $\mathbb{P}^2$, branched along a reduced curve of degree 8 as in the statement of Theorem 1 below.

A modern approach to Enriques surfaces has been carried out by Averbukh in [2, 15] and by Artin in [1]. The former one, in particular, showed again how to represent them as double planes. Equivalently, Enriques surfaces can be realized as double coverings of a quadric surface in $\mathbb{P}^3$, and these models have turned out to be very useful to study them, e.g. they allowed Horikawa to determine the periods of Enriques surfaces, see [14].

Nowadays, one usually says that Y is an Enriques surface if $q(Y) = 0$ and K_Y is a non-trivial element of 2-torsion in $\mathrm{Pic}(Y)$. In particular Y is supposed to be minimal. It is very well-known that Enriques surfaces form an irreducible family of dimension 10 and they are a distinguished class among surfaces with Kodaira dimension zero, which include also abelian, hyperelliptic and K3 surfaces. For a detailed account of the properties of Enriques surfaces, we refer the readers to the very interesting book [9] by Cossec and Dolgachev, where they considered Enriques surfaces in any characteristic; in particular see Chapter IV therein for a comprehensive report on their projective models (cf. also pp. 270–288 in [3]).

In this paper we present a short proof of the well-known representation of Enriques surfaces as double planes. Namely we will prove the following:

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THEOREM 1. *A smooth model of a double plane $\pi : X \to \mathbb{P}^2$ is a surface of Kodaira dimension zero with $q = p_g = 0$ if and only if there is a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced normal double plane, birationally equivalent to $\pi : X \to \mathbb{P}^2$, is branched along a reduced curve of degree 8 which has two lines L_1, L_2 as irreducible components and the residual sextic has the following singularities:*

1. *a double point at $p_0 = L_1 \cap L_2$;*
2. *a tacnode at a point $p_i \in L_i$, $i = 1, 2$, where L_i is the tacnodal tangent.*

Either p_1 or p_2 may possibly be infinitely near of the first order to p_0.

Let $\pi : X \to \mathbb{P}^2$ be a double plane and let $\rho : Y \to S$ be its canonical resolution, branched over the smooth curve B. One sees that, if Y has Kodaira dimension $-\infty$, then $|B + mK_S| = \emptyset$, for every $m \geq 2$, and in [5] we saw how to use these conditions in order to classify rational and ruled double planes.

If Y has Kodaira dimension zero and $p_g(Y) = q(Y) = 0$, i.e. Y is birationally equivalent to an Enriques surface, one sees that $p_a(B/2) = 0$, $|B/2 + K_S| = \emptyset$, $|B + mK_S| = \emptyset$ for $m > 2$ and $|B + 2K_S| = \{D\}$, where D is a curve which does not move (see Lemma 1 below). We will show that these conditions are enough to find a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as in the statement of Theorem 1.

In other words, our proof is based only on the properties of double covers and on the numerical characters (plurigenera and irregularity) of Enriques surfaces, with no need to use the geometry of curves on them.

In Section 1, we will fix notation and recall some well-known facts about double coverings. Then, in Section 2, we will prove Theorem 1.

Let us finally remark that a representation of Enriques surfaces as *fourfold* covers of $\mathbb{P}^2$ has been described by Verra in [16] and by Casnati and Ekedahl in [8].

1. Notation and preliminaries.

We consider algebraic varieties defined over the field of complex numbers $\mathbb{C}$. Let $\kappa(X)$ denote the Kodaira dimension of an algebraic variety X. A double plane $\pi : X \to \mathbb{P}^2$ is a double covering of the projective plane $\mathbb{P}^2$, i.e. π is a finite flat morphism of degree 2. Two double planes π and $\rho : Z \to \mathbb{P}^2$ are said to be *birationally equivalent* if there exists two birational maps $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and $\varphi : Z \dashrightarrow X$ such that $\pi \circ \varphi = \gamma \circ \rho$.

In particular, if X is normal, a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ uniquely determines the birational map $\varphi : Z \dashrightarrow X$, where Z is normal, and we will say that $\rho : Z \to \mathbb{P}^2$ is the double plane induced by π and γ.

Let us recall some well-known facts about double coverings (see, e.g., [3]). A double covering $\rho : Y \to S$ of any smooth rational surface S is uniquely determined by its branch curve C in S. Moreover C is smooth if and only if Y is smooth, and C is reduced if and only if Y is normal. If C is not reduced, say $C = \sum_i m_i C_i$, where the C_i's are the irreducible components of C and $m_i \geq 1$, then the normalization Y^ν of Y is a double covering of S branched over $\sum_i \varepsilon_i C_i$, where $\varepsilon_i = m_i \bmod 2 \in \{0, 1\}$.

Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C. If C is not smooth, there exists a birational morphism $\sigma : S \to \mathbb{P}^2$ such that the normalization Y of $X \times_{\mathbb{P}^2} S$ is smooth. The induced double covering $\rho : Y \to S$ is usually called *the canonical resolution* of π (see [3, p. 87] or [6]).

Let B be the branch curve of ρ and $\tilde{C}$ be the strict transform in S of C. Then $B = \tilde{C} + \sum_i \varepsilon_i E_i$, where $\varepsilon_i \in \{0, 1\}$ and the E_i's are the irreducible exceptional curves in S. Let us say that E_i is *branched* if $\varepsilon_i = 1$, and *unbranched* otherwise. Recall that B is an *even* divisor in S, i.e. $B/2$ is well-defined in the Picard group $\mathrm{Pic}(S)$ of S, and $\rho_*(\mathcal{O}_Y) \cong \mathcal{O}_S \oplus \mathcal{O}_S(-B/2)$, thus the *plurigenera* of Y are

$$P_m(Y) = h^0(S, mB/2 + mK_S) + h^0(S, (m-1)B/2 + mK_S),$$

for all $m \geq 1$, whereas its *irregularity* is $q(Y) = p_g(Y) - p_a(B/2)$.

In order to describe the singularities of C, it is convenient to recall and to use the classical notions of infinitely near points (cf. [13, p. 392], [12, v. 2, pp. 336–386], [7], or [5] in this setting). Let us write the birational morphism $\sigma : S \to \mathbb{P}^2$ as $\sigma = \sigma_n \circ \cdots \circ \sigma_1 \circ \sigma_0$, where $\sigma_i : S_i \to S_{i-1}$ is the blow-up at a point $x_i \in S_{i-1}$ and $\mathbb{P}^2 = S_{-1}$, $S = S_n$. One says that x_k is *infinitely near* to x_j, and we write $x_k > x_j$, if $x_k \in (\sigma_{k-1} \circ \cdots \circ \sigma_j)^{-1}(x_j)$. By $x_k >^s x_j$ we mean that x_k is infinitely near *of order s* to x_j. We say that x_k is *proper* if it is not infinitely near to x_j, for any $j \neq k$. In other words, a proper point really belongs to $\mathbb{P}^2$.

Let us denote by E_i (E_i^*, resp.) the strict (total, resp.) transform in S of the exceptional curve $\sigma_i^{-1}(x_i) \subset S_i$ of σ_i. Recall that $E_i = E_i^* - \sum_j q_{ij} E_j^*$ in $\mathrm{Pic}(S)$, where $q_{ij} \in \{0, 1\}$. One says that x_j is *proximate* to x_i if and only if $q_{ij} = 1$.

In $\mathrm{Pic}(S)$, write $\tilde{C} = 2dL - \sum_i c_i E_i^*$, where L is a total transform of a line, $2d = \tilde{C} \cdot L = \deg(C)$ and $c_i = \tilde{C} \cdot E_i^*$ is usually called the *multiplicity* of C at x_i. Then $B = \tilde{C} + \sum_i \varepsilon_i E_i = 2dL - \sum_i b_i E_i^*$, where $b_i = B \cdot E_i^* = c_i - \varepsilon_i + \sum_{j \neq i} \varepsilon_j q_{ji}$. Let us say that b_i is the *virtual* multiplicity of the branch curve of π at x_i.

Notice that if $x_k > x_j$, then $c_k \leq c_j$, because $\tilde{C} \cdot E_j \geq 0$. But the same is not true for the b_i's: it may happen that $x_k >^1 x_j$ and $b_k > b_j$. This occurs if and only if $b_k = b_j + 2$, $c_k = c_j$ and $\varepsilon_j = 1$. In that case, let us say that x_j (x_k, resp.) is a *defective* (*excessive*, resp.) point. One can check that x_j is defective if and only if E_i is a branched and $E_i^2 = -2$, or, equivalently, if and only if $\rho^{-1}(E_i)$ is a (-1)-curve in Y (see, e.g., [6] for more details).

For example, if C has a triple point $x_j \in \mathbb{P}^2$ with a triple point x_k infinitely near to it, i.e. in our notation $x_k >^1 x_j$ and $c_k = c_j = 3$, then $b_j = 2$, $\varepsilon_j = 1$ and $b_k = 4$, thus x_j is defective and x_k is excessive.

Regarding Cremona transformations $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, recall that Noether-Castelnuovo Theorem states that γ is the composition of finitely many *quadratic* Cremona transformations, i.e. such that the pull-back of the net of lines is a net of conics passing through three simple points, which can be proper or infinitely near. In particular, if these three points are x_0, x_1, x_2, with virtual multiplicity b_0, b_1, b_2, one checks that the branch curve of the induced normal double plane has degree $4d - b_0 - b_1 - b_2$ and virtual multiplicities $2d - b_1 - b_2$, $2d - b_0 - b_2$, $2d - b_0 - b_1$ at the points corresponding to x_0, x_1, x_2, respectively (cf., e.g., Lemma 5.1 in [5]).

2. Proof of Theorem 1.

First we determine some properties of the branch curve, and its adjoint linear systems, of a double plane whose canonical resolution is a surface Y of Kodaira dimension zero with $p_g = q = 0$. This clearly forces $P_{2n} = 1$ and $P_{2n+1} = 0$, for every $n \geq 1$, and the minimal model W of Y is such that $K_W^2 = 0$ (see, e.g., Lemma VIII.1 in [4]).

LEMMA 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane and $\rho : Y \to S$ its canonical resolution, branched over the smooth curve B in the smooth rational surface S. If Y is such that $\kappa(Y) = p_g(Y) = q(Y) = 0$, then*

(i) $|B/2 + K_S| = \emptyset$;

(ii) $p_a(B/2) = 0$;

(iii) $h^0(S, B + 2K_S) = 1$, *i.e.* $|B + 2K_S| = \{D\}$;

(iv) $|B + mK_S| = \emptyset$ *for* $m > 2$.

Proof. The double cover formulas for $p_g(Y)$ and $q(Y)$ recalled in §1 imply trivially (i) and (ii). If $m \geq 3$ is odd, say $m = 2n + 1$ with $n > 0$, then $P_m(Y) = 0$ forces $|nB + mK_S| = \emptyset$, therefore $|B + mK_S| = \emptyset$, because B is effective.

Since $P_2(Y) = 1$, one has either $|B + 2K_S| = \emptyset$ or $|B/2 + 2K_S| = \emptyset$, where the former (the latter, resp.) linear system corresponds to the invariant (anti-invariant, resp.) part of $|2K_Y|$. Note that the Riemann-Roch Theorem and $K_W^2 = 0$ imply that $h^0(-2K_W) > 0$, hence $\mathcal{O}_W(2K_W) \cong \mathcal{O}_W$. This means that the invariant part of $|2K_Y|$ is not empty, i.e. $|B+2K_S| = \{D\}$ is a curve which does not move. Since $P_{2n}(Y) = 1$, $n > 1$, it follows that $|B + 2nK_S| \subset |nB + 2nK_S| = \{nD\}$, and the inclusion is strict because B is effective and B cannot be part of D. Therefore $|B + 2nK_S| = \emptyset$, for $n > 1$, which concludes the proof. □

Now we want to show how to use the above properties (i)-(iv) in order to find a Cremona transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as in the statement of Theorem 1. This can be easily shown by applying the techniques we used to classify rational double planes. Indeed, the key results in [5] are Propositions 9.4 and 9.12, which can be stated together as follows:

PROPOSITION 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d$, and let $\rho : Y \to S$ be its canonical resolution, branched along the smooth curve B (cf. notation in §1). Suppose that $p_a(B/2) \geq -1$. If $|B+mK_S| = \emptyset$ for every $m \geq m_0$, where m_0 is a fixed integer with $m_0 \leq 2d/3$, then there exists a Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced double plane is branched along a curve of degree $2d'$ with a point x_0 of maximal virtual multiplicity $> 2(d' - m_0)$.* □

The main idea of the proof of the previous proposition is that the conditions $|B + mK_S| = \emptyset$, $m \geq m_0$, imply that the branch curve has singularities of "large" multiplicity at some points. This should imply that one can apply a quadratic Cremona transformation, centered at these points, which makes the branch curve somewhat "simpler", and then go on inductively. Proposition 9.4 in [5] shows that the following configuration of the singular points $x_0, \ldots, x_n$ of the branch curve is such that one does not easily see which quadratic Cremona transformation "simplifies" the branch curve:

($\star$) there is a point x_0 with $b_0 \geq 2(d - m_0)$ and each point x_i such that $b_i > d - b_0/2$, say for $i = 1, \ldots, h$, is excessive, say $x_i >^1 x_{h+i}$, with $b_i = 2 + d - b_0/2$ and such that there is a line L_i passing through x_0, x_i, x_{h+i}.

In this case, moreover, L_i is an irreducible component of the branch curve.

Proposition 9.12 in [5] shows that, if $p_a(B/2) \geq -1$, then configuration ($\star$) may occur only if $h = 3$ and $b_0 = b_1$, in which case one can apply two quadratic transformations centered at $x_1, \ldots, x_6$ and again one can "simplify" the branch curve.

In our situation Proposition 1 clearly implies the following:

COROLLARY 1. *Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d \geq 10$, and let $\rho : Y \to S$ be its canonical resolution. If Y is birationally equivalent to an Enriques surface, then there exists a Cremona transformation $\delta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced double plane is branched along a curve of degree $2d'$ with a point x_0 of maximal virtual multiplicity $b'_0 = 2d' - 4$.*

Proof. By Lemma 1, we can apply Proposition 1 with $m_0 = 3$. This implies the assertion with $b'_0 \geq 2d' - 4$. On the other hand, if $b'_0 \geq 2d' - 2$, then $\kappa(Y) = -\infty$ (cf., e.g., Lemma 8.6 in [5]) and we get a contradiction. □

Now we are ready to conclude the proof of Theorem 1.

Let $\pi : X \to \mathbb{P}^2$ be a normal double plane, branched along a reduced curve C of degree $2d$, with usual notation introduced in §1.

If $2d \leq 4$, then Y has Kodaira dimension $-\infty$.

Suppose that $2d = 6$. If the maximal virtual multiplicity is $b_0 \geq 4$, then again $\kappa(Y) = -\infty$. Otherwise, $b_0 \leq 2$ and $p_g(Y) = h^0(S, B/2 + K_S) = h^0(S, \mathcal{O}(S)) = 1$.

This forces $2d \geq 8$. Suppose that $2d = 8$. Again, if the maximal virtual multiplicity is $b_0 \geq 6$, then $\kappa(Y) = -\infty$. Let h be the number of points x_i with virtual multiplicity $b_i = 4$. Lemma 1, (ii), says that $0 = p_a(B/2) = 3 - h$, therefore $h = 3$. After re-ordering the indexes, we may assume that x_0, x_1, x_2 are the points with $b_0 = b_1 = b_2 = 4$.

Suppose that all of them are excessive, say $x_i >^1 x_{i+3}$, with $b_{i+3} = 2$, $i = 0, 1, 2$. Then we may assume that $x_3 \in \mathbb{P}^2$ and either $x_4 \in \mathbb{P}^2$ or $x_4 >^1 x_0$. In both cases the quadratic Cremona transformation centered at x_0, x_3, x_4 induces a normal double plane branched along a curve of degree 8 with a point, corresponding to x_1, which is not excessive and of virtual multiplicity 4.

So we may assume that $x_0 \in \mathbb{P}^2$. Note that, if we could find two points x_i and x_j with $b_i = 4$ and $b_j \geq 2$ such that there exists a quadratic Cremona transformation centered at x_0, x_i, x_j, then the induced normal double plane would be branched along a curve of degree ≤ 6, which contradicts our assumptions, according to the previous discussion.

This implies that both x_1, x_2 must be excessive, say $x_1 >^1 x_3$ and $x_2 >^1 x_4$, and moreover that there are two lines L_1, L_2 passing through x_0, x_3, x_1 and x_0, x_4, x_2, respectively. Note that this is configuration ($\star$) with $m_0 = h = 2$ and that

$$|B + 2K_S| = E_3 + E_4 + |2L - 2E_0^* - E_1^* - \cdots - E_4^*| = \{E_3 + E_4 + L_1 + L_2\},$$

which agrees with Lemma 1, (iii), where, abusing a little of notation, we denote by L_i also the strict transform in S of the line L_i, $i = 1, 2$. Note also that L_i is clearly also an irreducible component of the branch curve C, because it meets C at x_0, x_i, x_{i+2}, where C has multiplicity $c_0 = 4$, $c_i = 3$, $c_{i+2} = 3$, respectively. Setting $p_0 = x_0$ and $p_i = x_{i+2}$, $i = 1, 2$, this proves Theorem 1, in case $2d = 8$.

In order to conclude the proof of Theorem 1, it suffices to show that, if $2d \geq 10$, then there exists a Cremona transformation $\delta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the induced normal double plane has degree $< 2d$.

By Corollary 1, we know that $b_0 = 2d - 4$. Note that either

(i) $x_0 \in \mathbb{P}^2$, thus $c_0 \geq b_0 = 2d - 4$; or

(ii) there is no proper point of virtual multiplicity $2d - 4$ and x_0 is excessive, with $x_0 >^1 x_i \in \mathbb{P}^2$, for some i, thus $c_0 = c_i = 2d - 5$.

Consider first the latter case. Then $2d = 10$, otherwise the line $\overline{x_i x_0}$ would be a double component of the branch curve C, contradicting the assumption that C is reduced. Thus $b_0 = 6$, $b_i = 4$ and $c_0 = c_i = 5$. By Lemma 1, (i), we have that

$$\emptyset = |B/2 + K_S| = \overline{x_0 x_i} + E_i + |L - E_0^* - \cdots|$$

hence there is a point x_j with $b_j = 4$ such that either the quadratic Cremona transformation δ centered at x_0, x_i, x_j is well-defined, or $x_j >^1 x_k$, with $b_k \geq 2$, and the quadratic Cremona transformation δ' centered at x_0, x_i, x_k is well-defined. In both two situations, the branch curve of the induced normal double plane has degree < 10, which concludes the proof in case (ii).

Consider finally case (i). If there is a point x_i with $b_i \geq 6$, then apply a quadratic transformation centered at x_0, x_i and a general point x in the plane, thus the branch curve of the induced normal double plane has degree $\leq 2d - 2$ and the proof is done. So we may assume that, apart x_0, all other x_i's have $b_i \leq 4$. By Lemma 1, (ii), we have that $0 = p_a(B/2) = (d-1)(d-2)/2 - h$, where h is the number of points x_i, say $x_1, \ldots, x_h$, with $b_i = 4$.

We claim that there are two points x_i and x_j, with $b_i = 4$ and $b_j \geq 2$, such that the quadratic Cremona transformation centered at x_0, x_i, x_j is well-defined, therefore the branch curve of the induced normal double plane will have degree $\leq 2d-2$ and the proof of Theorem 1 will be concluded.

Indeed, either all the points $x_1, \ldots, x_h$ are excessive, or there is a point x_i, with $b_i = 4$, and such that either $x_i \in \mathbb{P}^2$ or $x_i >^1 x_0$. If all the x_i's are excessive, then $x_i >^1 x_{j(i)}$ with $b_{j(i)} = 2$, and moreover there is one of them, say x_k, such that either $x_{j(k)} \in \mathbb{P}^2$ or $x_{j(k)} >^1 x_0$.

Note that $x_1, \ldots, x_h$ cannot be all proximate to x_0, because C has multiplicity $2d-4$, or $2d-5$, at x_0, with $d > 4$, and $h = (d-1)(d-2)/2$. Thus we cannot find a quadratic transformation as above only if the points x_i are as in configuration ($\star$), with $m_0 = 2$. In that case, let L_i, $i = 1, \ldots, h$, be the strict transform in S of the line passing through x_0, x_{h+i}, x_i. For every $i = 1, \ldots, h$, the curve L_i should be a component of B and also of $|B + 2K_S|$, which is

$$|B+2K_S| = \sum_{i=h+1}^{h} E_{2h} + |(2d-6)(L-E_0^*) - \sum_{i=1}^{2h} E_i^*|$$

and we get a contradiction with Lemma 1, (iii), which says that $h^0(S, B+2K_S) = 1$, because we should have $h = (d-1)(d-2)/2$ such lines. □

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A REMARK ON THE AMPLE CONE OF $\overline{\mathcal{M}}_{g,n}$

Abstract. Here we address a question on the ample cone of the moduli spaces of curves with an inductive approach inspired by a paper of Arbarello and Cornalba.

1. Introduction

The moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g is a very natural and classical object, which from Riemann's times on has been deeply investigated by several authors. Despite this fact, the birational geometry of $\overline{\mathcal{M}}_{g,n}$ still remains a mystery. Indeed, every attempt in finding a regular pattern in the geometry of such a space seems destined to fail. This long history of trials and errors begins already in 1915 with Severi: being aware of the fact that $\mathcal{M}_g$ is unirational for $g \leq 10$, he was led to conjecture that $\mathcal{M}_g$ is unirational for all genera g (see [10], end of § 2.). That this is not really the case was shown in the first eighties by Eisenbud, Harris, and Mumford, who were able to prove that $\mathcal{M}_g$ is of general type for $g \geq 24$ (see [7], [2]). Anyway, the hope for a uniform description of the birational nature of the moduli spaces $\overline{\mathcal{M}}_g$ was still alive and inspired the celebrated Slope Conjecture by Harris and Morrison (see [6]): if $a\lambda - b\delta$ is the class of an effective divisor on $\overline{\mathcal{M}}_g$, then it should be $\frac{a}{b} \geq 6 + \frac{12}{g+1}$. In particular, since the class of the canonical divisor on $\overline{\mathcal{M}}_g$ is exactly $13\lambda - 2\delta$, it would follow that $\overline{\mathcal{M}}_g$ has negative Kodaira dimension for $g \leq 22$. Unfortunately, as recently pointed out by Farkas and Popa (see [4]), the Slope Conjecture does not hold for $g = 10$: a counterexample is provided by the divisor corresponding to curves on a K3 surface. Moreover, effective divisors behave in a wild manner already in genus zero: as observed by Keel and Vermeire (see [11]), the natural guess that every effective divisor on $\overline{\mathcal{M}}_{0,n}$ is an effective linear combination of boundary classes turns out to be false for every $n \geq 6$. However, one can still hope to fix at least the geometry of ample divisors. Recall that $\overline{\mathcal{M}}_{g,n}$ has a natural stratification by topological type, the codimension k strata corresponding to curves with at least k singular points. In the paper [5] by Gibney, Keel, and Morrison, the following Conjecture is attributed to Fulton:

CONJECTURE 1. ([5] (0.2)) A divisor on $\overline{\mathcal{M}}_{g,n}$ is ample if and only if it has positive intersection with all one-dimensional strata.

The main result of [5] is that Conjecture 1 holds for all g if and only if it holds for $g = 0$. In the same paper it is also described a natural approach to the case of $\overline{\mathcal{M}}_{0,n}$: as already pointed out by Keel and McKernan in [9], Conjecture 1 would be implied by a

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positive answer to:

QUESTION 1. ([5] (0.13)) If a divisor on $\overline{\mathcal{M}}_{0,n}$ has non-negative intersection with all one-dimensional strata, does it follow that the divisor is linearly equivalent to an effective combination of boundary divisors?

Until now, the best achievement in this direction is the following:

THEOREM 1. *For $n \leq 6$, the answer to Question 1 is affirmative.*

As we shall see, it is possible to approach such a result in several different ways. First of all, as noticed in [5] (0.14), Question 1 admits a purely combinatorial reformulation, which can be checked by using a computer precisely for $n \leq 6$ (indeed, its computational complexity makes it untractable already for $n = 7$). Next, a more conceptual analysis of the case $n = 6$ has been carried out in the paper [3] by Farkas and Gibney. Essentially, their idea is to express every divisor D on $\overline{\mathcal{M}}_{0,6}$ as an explicit linear combination of boundary divisors whose coefficients are intersection numbers with one-dimensional strata.

Here instead we are going to present a new proof, which follows an inductive strategy inspired by the paper [1] by Arbarello and Cornalba. Namely, let $P := \{1, 2, \ldots, n\}$ and for every $S \subset P$ with $2 \leq |S| \leq n-2$ let $\Delta_{\{0,S\}}$ be the boundary component of $\overline{\mathcal{M}}_{0,n}$ whose general element is the union of two copies of $\mathbb{P}^1$, labelled respectively by S and $P \setminus S$, meeting at one point. We denote by δ_S the corresponding class in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ and we define inductively:

$$\begin{aligned}
\mathscr{B}_4 &:= \{\delta_{\{2,3\}}\} \\
\mathscr{B}_n &:= \mathscr{B}_{n-1} \cup \{\delta_B : B \subseteq \{1, \ldots, n\}, n \notin B \supseteq \{n-1, n-2\}\} \\
&\quad \cup \{\delta_{B^c \setminus \{n\}} : \delta_B \in \mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}\}.
\end{aligned}$$

Then we have

PROPOSITION 1. *$\mathscr{B}_n$ is a basis of* $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$.

Moreover, for $n \leq 5$ every divisor having non-negative intersection with all one-dimensional strata can be expressed as an effective linear combination of divisors in $\mathscr{B}_n$; for $n = 6$ this is no longer the case, but one still maintains a control on the sign of coefficients, which is strong enough to conclude the proof of Theorem 1 in a few lines. Unfortunately, as n grows up, the combinatorial complexity of the problem explodes: indeed, even the case $n = 7$ seems to be completely out of reach.

2. The tools

We are going to make essential use of the following basic facts:

LEMMA 1. *(Arbarello–Cornalba) Let* $\vartheta : \overline{\mathcal{M}}_{0,A\cup\{q\}} \longrightarrow \overline{\mathcal{M}}_{0,P}$ *be the map which associates to any* $A \cup \{q\}$*-pointed genus zero curve the* P*-pointed genus zero curve obtained by glueing to it a fixed* $A^c \cup \{r\}$*-pointed genus zero curve via identification of* q *and* r*. Then for every* $B \subset P$ *with* $B \neq A$ *and* $B \neq A^c$ *we have*

$$\vartheta^*(\delta_B) = \begin{cases} \delta_B & \textit{if } B \subset A \textit{ and } B \neq A \\ \delta_{B\setminus A^c \cup\{q\}} & \textit{if } B \supset A^c \textit{ and } B \neq A^c \\ 0 & \textit{otherwise} \end{cases}$$

(see [1], Lemma 3.3).

LEMMA 2. *(Keel) If* $a, b, c, d \in \{1, 2, \ldots, n\}$ *are four distinct elements, then the following relation holds in* $\overline{\mathcal{M}}_{0,n}$*:*

$$\sum_{\substack{a,b\in T\\ c,d\notin T}} \delta_T = \sum_{\substack{a,c\in T\\ b,d\notin T}} \delta_T$$

(see [8], (2) p. 550).

LEMMA 3. *(Gibney-Keel-Morrison) Let*

$$D = \sum_{|S|\geq 2} b_S \delta_S$$

be a divisor on $\overline{\mathcal{M}}_{0,n}$ *and set* $b_S := 0$ *for* $|S| = 1$*. Then* D *has non-negative intersection with all one-dimensional strata if and only if*

$$b_{I\cup J} + b_{I\cup K} + b_{I\cup L} \geq b_I + b_J + b_K + b_L$$

for every partition $I \cup J \cup K \cup L = \{1, 2, \ldots, n\}$

(see [5], Theorem 2.1).

For further details, we refer the interested reader to the original papers; however, we stress that the corresponding proofs are very short and elementary.

3. The proofs

Proof of Proposition 1. From [8] it is known that $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ is a free group on $2^{n-1} - \binom{n}{2} - 1$ generators. Therefore, in order to get the claim it will be sufficient to show:

(1) $$|\mathscr{B}_n| = 2^{n-1} - \binom{n}{2} - 1;$$

(2) there are no linear relations among the elements of $\mathscr{B}_n$.

We are going to check (1) by induction on n.

If $n = 4$, it is clear that $\mathscr{B}_4$ has the right cardinality.
If $n \geq 5$, by inductive assumption we have

$$|\mathscr{B}_{n-1}| = 2^{n-2} - \binom{n-1}{2} - 1$$

and

$$|\mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}| = \sum_{k=0}^{n-5} \binom{n-3}{k}.$$

It follows that

$$\begin{aligned} |\mathscr{B}_n \setminus \mathscr{B}_{n-1}| &= \sum_{k=0}^{n-4} \binom{n-3}{k} + |\mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}| = \\ &= \sum_{k=0}^{n-4} \left[\binom{n-3}{k} + \binom{n-3}{k-1} \right] = \sum_{k=0}^{n-4} \binom{n-2}{k} \end{aligned}$$

and

$$\begin{aligned} 2^{n-1} - \binom{n}{2} - 1 &= 2^{n-2} + \sum_{k=0}^{n-2} \binom{n-2}{k} - \binom{n-1}{2} - (n-1) - 1 = \\ &= 2^{n-2} - \binom{n-1}{2} - 1 + \sum_{k=0}^{n-4} \binom{n-2}{k} = |\mathscr{B}_n|. \end{aligned}$$

As for (2), let us argue by induction on n again.
If $n = 4$ there is nothing to prove.
If $n \geq 5$, let $\sum a_B \delta_B = 0$ be a linear relation in $\mathscr{B}_n$. By Lemma 1 applied to $A := P \setminus \{n, n-1\}$ we have:

$$0 = \vartheta^*(\sum a_B \delta_B) = \sum_{\delta_B \in \mathscr{B}_{n-1}} a_B \delta_B$$

hence by inductive assumption $a_B = 0$ for every $\delta_B \in \mathscr{B}_{n-1}$. Next, by Lemma 1 applied to $A := P \setminus \{n, n-2\}$ we have:

$$0 = \vartheta^*(\sum a_B \delta_B) = \sum_{\delta_{B^c \setminus \{n\}} \in \mathscr{B}_{n-1}} a_B \delta_B$$

hence by inductive assumption $a_B = 0$ for every δ_B such that $\delta_{B^c \setminus \{n\}} \in \mathscr{B}_{n-1}$. In order to conclude, we have only to show that the elements in $\mathscr{B}_n$ with $n \notin B \supseteq \{n-1, n-2\}$ are linearly independent. This fact is a direct consequence of [1], Lemma 3.9, so the proof is over.

□

Proof of Theorem 1. The case $n = 4$ is obvious. Fix $n = 5$ and let D be a divisor on $\overline{\mathcal{M}}_{0,5}$ having non-negative intersection with all one-dimensional strata. Write

$$D = c_{\{2,3\}}\delta_{\{2,3\}} + c_{\{3,4\}}\delta_{\{3,4\}} + c_{\{1,5\}}\delta_{\{1,5\}} + c_{\{2,5\}}\delta_{\{2,5\}} + c_{\{1,4\}}\delta_{\{1,4\}}$$

in the basis $\mathscr{B}_5$. From Lemma 3 it follows that $c_{\{2,3\}} \geq 0$ (let $I = \{2\}$, $J = \{3\}$, $K = \{1\}$, $L = \{4,5\}$); $c_{\{3,4\}} \geq 0$ (let $I = \{3\}$, $J = \{4\}$, $K = \{5\}$, $L = \{1,2\}$); $c_{\{1,5\}} \geq 0$ (let $I = \{1\}$, $J = \{5\}$, $K = \{3\}$, $L = \{2,4\}$); $c_{\{2,5\}} \geq 0$ (let $I = \{2\}$, $J = \{5\}$, $K = \{4\}$, $L = \{1,3\}$); $c_{\{1,4\}} \geq 0$ (let $I = \{1\}$, $J = \{4\}$, $K = \{2\}$, $L = \{3,5\}$). Hence the case $n = 5$ is over. Fix now $n = 6$, let D be a divisor on $\overline{\mathcal{M}}_{0,6}$ having non-negative intersection with all one-dimensional strata and express $D = \sum c_B\delta_B$ in the basis $\mathscr{B}_6$. ¿From Lemma 1 applied to $A := P \setminus \{6,5\}$ and to $A := P \setminus \{6,4\}$ as in the proof of Proposition 1 and from Lemma 3 it follows that all coefficients are non-negative, with the unique possible exception of $\delta_{\{1,4,5\}}$. However, if $c_{\{1,4,5\}} < 0$, then by applying the relation:

$$\sum_{\substack{4,5 \in B \\ 2,3 \notin B}} \delta_B = \sum_{\substack{2,5 \in B \\ 3,4 \notin B}} \delta_B$$

(see Lemma 2), we can replace $\delta_{\{1,4,5\}}$ with $\delta_{\{1,2,3\}}$. If we express

$$D = \sum_{B \neq \{1,4,5\}} c'_B\delta_B + c'_{\{1,2,3\}}\delta_{\{1,2,3\}}$$

we have $c'_B \neq c_B$ only for $B = \{3,4\}$, $B = \{1,3,4\}$, $B = \{2,5\}$, and $B = \{1,2,5\}$; in all these cases, Lemma 1 shows that $c'_B \geq 0$. Since $c_B \geq 0$ for every $B \neq \{1,2,3\}$ and either $c_{\{1,4,5\}}$ or $c'_{\{1,2,3\}}$ is non-negative, it follows that Question 1 has a positive answer also for $n = 6$.

□

REMARK 1. In the case $n = 6$, one may wonder whether the sign of $c_{\{1,4,5\}}$ is actually ambiguous or not. Indeed, it is possible to construct explicit examples with $c_{\{1,4,5\}} < 0$ (for instance, take $c_{\{1,4,5\}} = -1$, $c_{\{2,3,4\}} = c_{\{1,2,5\}} = c_{\{3,4\}} = c_{\{2,3,5\}} = c_{\{1,3,4\}} = c_{\{2,5\}} = c_{\{2,4,5\}} = c_{\{3,4,5\}} = c_{\{2,3,4,5\}} = 1$, $c_{\{1,5\}} = c_{\{2,3\}} = c_{\{1,2,4,5\}} = c_{\{1,3,4,5\}} = c_{\{1,4\}} = c_{\{4,5\}} = 0$ and use Lemma 3) and with $c_{\{1,4,5\}} > 0$ (for instance, take $c_{\{1,4,5\}} = c_{\{4,5\}} = c_{\{2,3\}} = c_{\{2,3,4,5\}} = c_{\{2,3,4\}} = c_{\{2,3,5\}} = c_{\{2,4,5\}} = c_{\{3,4,5\}} = 1$, $c_{\{1,2,5\}} = c_{\{3,4\}} = c_{\{1,3,4\}} = c_{\{2,5\}} = c_{\{1,5\}} = c_{\{1,2,4,5\}} = c_{\{1,3,4,5\}} = c_{\{1,4\}} = 0$ and use Lemma 3). We also point out that the number of indeterminate signs grows up with n (for instance, in the case $n = 7$, none of the coefficients $c_{\{1,4,5\}}$, $c_{\{2,3,6\}}$, $c_{\{1,2,5,6\}}$, $c_{\{2,5,6\}}$, $c_{\{2,3,5,6\}}$, and $c_{\{1,4,5,6\}}$ is forced to be non-negative by Lemma 3); moreover, for $n \geq 7$ there seems to be no uniform way to apply a relation from Lemma 2 in order to remove a negative sign without introducing any other one.

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Polynomial Interpolation and Projective Embeddings

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A NEW INVARIANT FOR PLANE CURVE SINGULARITIES

Abstract. In [5] the authors gave a general sufficient numerical condition for the T-smoothness (smoothness and expected dimension) of equisingular families of plane curves. This condition involves a new invariant γ^* for plane curve singularities, and it is conjectured to be asymptotically proper. In [9], similar sufficient numerical conditions are obtained for the T-smoothness of equisingular families on various classes surfaces. These conditions involve a series of invariants γ^*_α, $0 \leq \alpha \leq 1$, with $\gamma^*_1 = \gamma^*$. In the present paper we compute (respectively give bounds for) these invariants for semiquasihomogeneous singularities.

When studying numerical conditions for the T-smoothness of equisingular families of curves, new invariants of plane curve singularities $V(f) \subset (\mathbb{C}^2, 0)$ turn up. These invariants are defined as the maximum of a function depending on the codimension of complete intersection ideals containing the Tjurina ideal, respectively the equisingularity ideal, of f, and on the intersection multiplicity of f with elements of the complete intersection ideals. In Section 1 we will define these invariants, and we will calculate them for several classes of singularities, the main results being Proposition 1, Proposition 2 and Proposition 3. It is the upper bound in Lemma 3 which ensures that the conditions for T-smoothness with these new conditions (see [4], [5], [9]) improve the previously known ones (see [3]). In the remaining sections we introduce some notation and we gather some necessary, though mainly well-known technical results used in the proofs of Section 1.

We should like to point out that the definition of the invariant γ_1^* below is a modification of the invariant "γ^*" defined in [5], and it is always bound from above by the latter. Moreover, the latter can be replaced by it in the conditions of [5] Proposition 2.2.

NOTATION 1. Throughout this paper, $R = \mathbb{C}\{x, y\}$ will be the ring of convergent power series in the variables x and y, and $\mathfrak{m} = \langle x, y\rangle \lhd R$ will be its maximal ideal.

1. The γ^*_α-invariants

For the definition of the γ^*_α-invariants the Tjurina ideal, respectively the equisingularity ideal in the sense of [12], play an essential role. For the convenience of the reader we recall their definitions.

DEFINITION 1. *Let* $f \in \mathfrak{m}$ *be a reduced power series. The* Tjurina ideal *of* f

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is defined as

$$I^{ea}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle,$$

and the equisingularity ideal *of* f *is defined as*

$$I^{es}(f) = \left\{ g \in R \;\middle|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \right\} \supseteq I^{ea}(f).$$

Their codimensions

$$\tau(f) = \dim_{\mathbb{C}} R/I^{ea}(f),$$

respectively

$$\tau^{es}(f) = \dim_{\mathbb{C}} R/I^{es}(f),$$

are analytical, respectively topological, invariants of the singularity type defined by f*. Note that* $\tau^{es}(f)$ *is the codimension of the* μ*-constant stratum in the equisingular deformation of the plane curve singularity defined by* f*. It can be computed in terms of multiplicities of the strict transform of* f *at essential infinitely near points in the resolution tree of* $\big(V(f), 0\big)$ *(cf. [10]).*

DEFINITION 2. *Let* $f \in \mathfrak{m}$ *be a reduced power series, and let* $0 \leq \alpha \leq 1$ *be a rational number.*
If I *is a zero-dimensional ideal in* R *with* $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ *and* $g \in I$*, we define*

$$\lambda_\alpha(f; I, g) := \frac{\big(\alpha \cdot i(f,g) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)},$$

and

$$\gamma_\alpha(f; I) := \max \Big\{ (1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\Big|\; g \in I, i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I) \Big\},$$

where $i(f,g)$ *denotes the intersection multiplicity of* f *and* g*. Note that, by Lemma 1,* $i(f,g) > \dim_{\mathbb{C}}(R/I)$ *for all* $g \in I$*. Thus* $\gamma_\alpha(f; I)$ *is a well-defined positive rational number.*
We then set

$$\gamma_\alpha^{ea}(f) := \max \left\{ 0,\ \gamma_\alpha(f; I) \;\middle|\; I \supseteq I^{ea}(f) \text{ is a complete intersection ideal} \right\}$$

and

$$\gamma_\alpha^{es}(f) := \max \left\{ 0,\ \gamma_\alpha(f; I) \;\middle|\; I \supseteq I^{es}(f) \text{ is a complete intersection ideal} \right\}$$

Note, if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$*, then* $I^{ea}(f) = I^{es}(f) = R$ *and there is no zero-dimensional complete intersection ideal containing them, hence* $\gamma_\alpha^{ea}(f) = \gamma_\alpha^{es}(f) = 0$*.*

LEMMA 1. *Let* $f \in \mathfrak{m}^2$ *be reduced, and let* I *be an ideal such that* $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$*.*
Then, for any $g \in I$*, we have*

$$\dim_{\mathbb{C}}(R/I) < \dim_{\mathbb{C}} \big(R/\langle f, g\rangle\big) = i(f,g).$$

Proof. Cf. [11] Lemma 4.1; the idea is mainly to show that not both derivatives of f can belong to $\langle f, g\rangle$. □

Up to embedded isomorphism the Tjurina ideal only depends on the analytical type of the singularity. More precisely, if $f \in R$ is any power series, $u \in R$ a unit and $\phi : R \to R$ an isomorphism, then $I^{ea}(u \cdot f \circ \phi) = \{g \circ \phi \mid g \in I^{ea}(f)\}$. Thus the following definition makes sense.

DEFINITION 3. *Let $\mathcal{S}$ be an analytical, respectively topological, singularity type, and let $f \in R$ be a representative of $\mathcal{S}$. We then define*

$$\gamma_\alpha^{ea}(\mathcal{S}) := \gamma_\alpha^{ea}(f),$$

respectively

$$\gamma_\alpha^{es}(\mathcal{S}) := \max\{\gamma_\alpha^{es}(g) \mid g \text{ is a representative of } \mathcal{S}\}.$$

Since $i(f, g) > \dim_{\mathbb{C}}(R/I)$ in the above situation, we deduce the following lemma.

LEMMA 2. *Let $f \in \mathfrak{m}^2$ be reduced, $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$ be a zero-dimensional ideal, and $0 \leq \alpha < \beta \leq 1$, then $\gamma_\alpha(f; I) < \gamma_\beta(f; I)$.*
In particular, for any analytical, respectively topological singularity type

$$\gamma_\alpha^{ea}(\mathcal{S}) < \gamma_\beta^{ea}(\mathcal{S}) \qquad \text{respectively,} \qquad \gamma_\alpha^{es}(\mathcal{S}) < \gamma_\beta^{es}(\mathcal{S}).$$

For reasons of comparison let us also recall the definition of τ_{ci}^{ea}, τ_{ci}^{es}, κ and δ.

DEFINITION 4. *For $f \in R$ we define*

$$\tau_{ci}^{ea}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{ea}(f) \text{ a complete intersection}\},$$

and

$$\tau_{ci}^{es}(f) := \max\{0, \dim_{\mathbb{C}}(R/I) \mid I \supseteq I^{es}(f) \text{ a complete intersection}\}.$$

Again, for analytically equivalent singularities the values coincide, so that for an analytical singularity type $\mathcal{S}$, choosing some representative $f \in R$, we may define

$$\tau_{ci}^{ea}(\mathcal{S}) := \tau_{ci}(f).$$

For a topological singularity type we set

$$\tau_{ci}^{es}(\mathcal{S}) := \max\{\tau_{ci}^{es}(g) \mid g \text{ a representative of } \mathcal{S}\}.$$

Note that obviously

$$\tau_{ci}^{ea}(\mathcal{S}) \leq \tau(\mathcal{S}) \quad \text{and} \quad \tau_{ci}^{es}(\mathcal{S}) \leq \tau^{es}(\mathcal{S}),$$

where $\tau(\mathcal{S})$ is the Tjurina number of $\mathcal{S}$ and $\tau^{es}(\mathcal{S})$ is as defined in Definition 1.

DEFINITION 5. *For $f \in R$ and $\mathcal{O} = R/\langle f \rangle$, we define the δ-invariant*

$$\delta(f) = \dim_{\mathbb{C}} \widetilde{\mathcal{O}}/\mathcal{O}$$

where $\mathcal{O} \subset \widetilde{\mathcal{O}}$ is the normalisation of $\mathcal{O}$, and the κ-invariant

$$\kappa(f) = i\left(f, \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial x}\right),$$

where $(\alpha : \beta) \in \mathbb{P}^1_{\mathbb{C}}$ is generic.

δ and κ are topological (thus also analytical) invariants of the singularity defined by f so that for the topological, respectively analytical, singularity type $\mathcal{S}$ given by f we can set

$$\delta(\mathcal{S}) = \delta(f) \quad \text{and} \quad \kappa(\mathcal{S}) = \kappa(f).$$

Throughout this article we will sometimes treat topological and analytical singularities at the same time. Whenever we do so, we will write $I^*(f)$ for $I^{ea}(f)$ respectively, for $I^{ea}(f)$, and analogously we will use the notation γ^*_α, τ^*_{ci} and τ^*.

The following lemma is again obvious from the definition of $\gamma_\alpha(f; I)$, once we take into account that $\kappa(f) = i(f, g)$ for a generic element $g \in I^{ea}(f)$ of f and that for a fixed value of $d = \dim_{\mathbb{C}}(R/I)$ the function $i \mapsto \frac{(\alpha i + (1-\alpha)\cdot d)^2}{i-d}$ takes its maximum on $[d+1, 2d]$ for the minimal possible value $i = d+1$.

LEMMA 3. *Let $f \in \mathfrak{m}^2$ be reduced, and let I be an ideal in R such that $I^{ea}(f) \subseteq I \subseteq \mathfrak{m}$.*
Then

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \gamma_\alpha(f; I) \leq \big(\dim_{\mathbb{C}}(R/I) + \alpha\big)^2.$$

Moreover, if $\kappa(f) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$, then

$$\gamma_\alpha(f; I) \geq \frac{\big(\alpha \cdot \kappa(f) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{\kappa(f) - \dim_{\mathbb{C}}(R/I)}.$$

In particular, for any analytical, respectively topological, singularity type $\mathcal{S}$

$$(1+\alpha)^2 \cdot \tau^*_{ci}(\mathcal{S}) \leq \gamma^*_\alpha(\mathcal{S}) \leq \big(\tau^*_{ci}(\mathcal{S}) + \alpha\big)^2,$$

and if $\kappa(\mathcal{S}) \leq 2 \cdot \tau^*_{ci}(\mathcal{S})$, then

$$\gamma^*_\alpha(\mathcal{S}) \geq \frac{\big(\alpha \cdot \kappa(\mathcal{S}) + (1-\alpha) \cdot \tau^*_{ci}(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau^*_{ci}(\mathcal{S})}.$$

In order to make the conditions for T-smoothness in [9] as sharp as possible, it is useful to know under which circumstances the term $(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I)$ involved in the definition of $\gamma^*_\alpha(\mathcal{S})$ is actually exceeded.

LEMMA 4. *If $\mathcal{S}$ is a topological or analytical singularity type such that $\kappa(\mathcal{S}) < 2\cdot\tau^*_{ci}(\mathcal{S})$, then*

$$(1+\alpha)^2 \cdot \tau^*_{ci}(\mathcal{S}) < \gamma^*_\alpha(\mathcal{S}).$$

This is in particular the case, if $\mathcal{S} \neq A_1$ and $\tau^*_{ci}(\mathcal{S}) = \tau^*(\mathcal{S})$, i. e. if the Tjurina ideal, respectively the equisingularity ideal, of some representative is a complete intersection.

Proof. Lemma 3 gives

$$\gamma^*_\alpha(\mathcal{S}) \geq \frac{\big(\alpha\cdot\kappa(\mathcal{S}) + (1-\alpha)\cdot\tau^*_{ci}(\mathcal{S})\big)^2}{\kappa(\mathcal{S}) - \tau^*_{ci}(\mathcal{S})}.$$

If we consider the right-hand side as a function in $\kappa(\mathcal{S})$, it is strictly decreasing on the interval $\big[0, 2\cdot\tau^*_{ci}(\mathcal{S})\big]$ and takes its minimum thus at $2\cdot\tau^*_{ci}(\mathcal{S})$. By the assumption on $\kappa(\mathcal{S})$ we, therefore, get

$$\gamma^*_\alpha(\mathcal{S}) > (1+\alpha)^2 \cdot \tau^*_{ci}(\mathcal{S}).$$

Suppose now that $\tau^*_{ci}(\mathcal{S}) = \tau^*(\mathcal{S})$ and $\mathcal{S} \neq A_1$. By Lemma 5 we know $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S}) \leq \tau(\mathcal{S})$. On the other hand, we have $\kappa(\mathcal{S}) \leq 2\cdot\delta(\mathcal{S})$ (see [6]). Therefore, $\kappa(\mathcal{S}) < 2\cdot\tau^*_{ci}(\mathcal{S})$. $\square$

LEMMA 5. *If $\mathcal{S} \neq A_1$ is any analytical or topological singularity type, then $\delta(\mathcal{S}) < \tau^{es}(\mathcal{S})$.*

Proof. If (C,z) is a representative of $\mathcal{S}$ and if $\mathcal{T}^*(C,z)$ is the essential subtree of the complete embedded resolution tree of (C,z), then

$$\delta(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\operatorname{mult}_p(C)\cdot(\operatorname{mult}_p(C)-1)}{2}$$

and

$$\tau^{es}(\mathcal{S}) = \sum_{p\in\mathcal{T}^*(C,z)} \frac{\operatorname{mult}_p(C)\cdot(\operatorname{mult}_p(C)+1)}{2} - \#\text{ free points in } \mathcal{T}^*(C,z) - 1,$$

where $\operatorname{mult}_p(C)$ denotes the multiplicity of the strict transform of C at p (see [6]). Setting $\varepsilon_p = 0$ if p is satellite, $\varepsilon_p = 1$ if $p \neq z$ is free, and $\varepsilon_z = 2$, then $\operatorname{mult}_p(C) \geq \varepsilon_p$ and therefore

$$\tau^{es}(\mathcal{S}) = \delta(\mathcal{S}) + \sum_{p\in\mathcal{T}^*(C,z)} \big(\operatorname{mult}_p(C) - \varepsilon_p\big) \geq \delta(\mathcal{S}).$$

Moreover, we have equality if and only if $\operatorname{mult}_z(C) = 2$, $\operatorname{mult}_p(C) = 1$ for all $p \neq z$ and there is no satellite point, but this implies that $\mathcal{S} = A_1$. $\square$

For some classes of singularities we can calculate the γ_α^*-invariant concretely, and for some others we can at least give an upper bound, which in general is much better than the one derived from Lemma 3. We restrict our attention to singularities having a convenient semi-quasihomogeneous representative $f \in R$ (see Definition 8). Throughout the following proofs we will frequently make use of monomial orderings, see Section 2.

PROPOSITION 1 (SIMPLE SINGULARITIES). *Let α be a rational number with $0 \leq \alpha \leq 1$. Then we obtain the following values for $\gamma_\alpha^{es}(\mathcal{S}) = \gamma_\alpha^{ea}(\mathcal{S})$, where $\mathcal{S}$ is a simple singularity type.*

$\mathcal{S}$		$\gamma_\alpha^{ea}(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$
A_k,	$k \geq 1$	$(k+\alpha)^2$
D_k,	$4 \leq k \leq 4+\sqrt{2}\cdot(2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
D_k,	$k \geq 4+\sqrt{2}\cdot(2+\alpha)$	$(k-2+\alpha)^2$
E_k,	$k = 6, 7, 8$	$\frac{(k+2\alpha)^2}{2}$

Proof. Let $\mathcal{S}_k$ be one of the simple singularity types A_k, D_k or E_k, and let $f \in R$ be a representative of $\mathcal{S}_k$. Note that the Tjurina ideal $I^{ea}(f)$ and the equisingularity ideal $I^{es}(f)$ coincide, and hence so do the γ_α^*-invariants, i. e.

$$\gamma_\alpha^{ea}(\mathcal{S}_k) = \gamma_\alpha^{es}(\mathcal{S}_k).$$

Moreover, in the considered cases the Tjurina ideal is indeed a complete intersection ideal with $\dim_{\mathbb{C}}\big(R/I^{ea}(f)\big) = k$, so that in particular the given values are upper bounds for $(1+\alpha)^2\cdot\dim_{\mathbb{C}}(R/I)$ for any complete intersection ideal I containing the Tjurina ideal. By Lemma 3 we know

$$\frac{(\alpha\cdot\kappa(\mathcal{S}_k)+(1-\alpha)\cdot k)^2}{\kappa(\mathcal{S}_k)-k} \leq \gamma_\alpha(\mathcal{S}_k) \leq (k+\alpha)^2.$$

Note that $\kappa(A_k) = k+1$, $\kappa(D_k) = k+2$ and $\kappa(E_k) = k+2$, which in particular gives the result for $\mathcal{S}_k = A_k$. Moreover, it shows that for $\mathcal{S}_k = D_k$ or $\mathcal{S}_k = E_k$ we have

$$\gamma_\alpha(\mathcal{S}_k) \geq \frac{(k+2\alpha)^2}{2}.$$

If we fix a complete intersection ideal I with $I^{ea}(f) \subseteq I$, then

$$\lambda_\alpha(f; I, g) = \frac{\big(\alpha\cdot i(f,g)+(1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{i(f,g)-\dim_{\mathbb{C}}(R/I)},$$

with $g \in I$ such that $i(f,g) \leq 2\cdot\dim_{\mathbb{C}}(R/I)$, considered as a function in $i(f,g)$ is maximal, when $i(f,g)$ is minimal. If $i(f,g)-\dim_{\mathbb{C}}(R/I) \geq 2$, then

$$\lambda_\alpha(f; I, g) \leq \frac{(k+2\alpha)^2}{2}.$$

It therefore remains to consider the case where

$$i(f,g) - \dim_{\mathbb{C}}(R/I) = 1 \tag{1}$$

for some I and some $g \in I$, and to maximise the possible $\dim_{\mathbb{C}}(R/I)$.

We claim that for $\mathcal{S}_k = D_k$ with $f = x^2y - y^{k-1}$ as representative, $\dim_{\mathbb{C}}(R/I) \leq k-2$, and thus $I = \langle x, y^{k-2}\rangle$ and $g = x$ are suitable with

$$\lambda_\alpha(f; I, x) = (k-2+\alpha)^2,$$

which is greater than $\frac{(k+2\alpha)^2}{2}$ if and only if $k \geq 4 + \sqrt{2}\cdot(2+\alpha)$. Suppose, therefore, $\dim_{\mathbb{C}}(R/I) = k-1$. Then $y^{k-1}, x^3 \in I^{ea}(f) = \langle xy, x^2 - (k-1)\cdot y^{k-2}\rangle \subset I$, the leading ideal $L_{<_{ls}}\big(I^{ea}(f)\big) = \langle x^3, xy, y^{k-2}\rangle \subset L_{<_{ls}}(I)$, and since by Proposition 4 $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_{<_{ls}}(I)\big)$, either $L_{<_{ls}}(I) = \langle x^3, xy, y^{k-3}\rangle$ or $L_{<_{ls}}(I) = \langle x^2, xy, y^{k-2}\rangle$. In the first case there is a power series $g \in I$ such that $g \equiv y^{k-3} + ax + bx^2 \pmod{I}$, and hence $I \ni yg \equiv y^{k-2} \pmod{I}$, i. e. $y^{k-2} \in I$. But then $x^2 \in I$ and $x^2 \in L_{<_{ls}}(I)$, in contradiction to the assumption. In the second case, similarly, there is a $g \in I$ such that $g \equiv x^2 \pmod{I}$, and hence $x^2 \in I$ which in turn implies that $y^{k-2} \in I$. Thus $I = \langle x^2, xy, y^{k-2}\rangle$, and $\dim_{\mathbb{C}}(I/\mathfrak{m}I) = 3$ which by Remark 8 contradicts the fact that I is a complete intersection.

The cases of the exceptional singularities E_6, E_7 and E_8 are treated similarly. □

PROPOSITION 2 (ORDINARY MULTIPLE POINTS). *Let α be a rational number with $0 \leq \alpha \leq 1$, and let M_k denote the topological singularity type of an ordinary k-fold point with $k \geq 3$. Then*

$$\gamma_\alpha^{es}(M_k) = 2\cdot(k-1+\alpha)^2.$$

In particular

$$\gamma_\alpha^{es}(M_k) > (1+\alpha)^2 \cdot \tau_{ci}^{es}(M_k).$$

Proof. Note that for any representative f of M_k we have

$$I^{es}(f) = I^{ea}(f) + \mathfrak{m}^k = \left\langle \frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}\right\rangle + \mathfrak{m}^k,$$

where f_k is the homogeneous part of degree k of f, so that we may assume f to be homogeneous of degree k.

If I is a complete intersection ideal with $\mathfrak{m}^k \subset I^{es}(f) \subseteq I$, then by Lemma 9

$$\dim_{\mathbb{C}}(R/I) \leq \big(k - \mathrm{mult}(I) + 1\big)\cdot \mathrm{mult}(I).$$

We note moreover that for any $g \in I$

$$i(f,g) \geq \mathrm{mult}(f)\cdot\mathrm{mult}(g) \geq k\cdot\mathrm{mult}(I),$$

and that for a fixed I we may attain an upper bound for $\lambda_\alpha(f;I,g)$ by replacing $i(f,g)$ by a lower bound for $i(f,g)$.

Hence, if $\mathrm{mult}(I) \geq 2$, we have

$$(2) \quad \lambda_\alpha(f;I,g) \leq \frac{\big(k-(1-\alpha)\cdot(\mathrm{mult}(I)-1)\big)^2 \cdot \mathrm{mult}(I)^2}{\mathrm{mult}(I)\cdot\big(\mathrm{mult}(I)-1\big)} \leq 2\cdot(k-1+\alpha)^2,$$

while $\dim_{\mathbb{C}}(R/I) \leq k-1$ for $\mathrm{mult}(I)=1$ and the above inequality (2) is still satisfied. This together with Lemma 9 shows

$$\gamma_\alpha^{es}(M_k) \leq 2\cdot(k-1+\alpha)^2.$$

On the other hand, considering the representative $f = x^k - y^k$, we have

$$I^{es}(f) = \langle x^{k-1}, y^{k-1}, x^a y^b \mid a+b=k\rangle,$$

and $I = \langle y^{k-1}, x^2\rangle$ is a complete intersection ideal containing $I^{es}(f)$. Moreover, $i\big(f,x^2\big) = 2k$, $\dim_{\mathbb{C}}(R/I) = 2\cdot(k-1)$, thus

$$\gamma_\alpha^{es}(M_k) \geq \frac{\big(\alpha\cdot i(f,x^2)+(1-\alpha)\cdot\dim_{\mathbb{C}}(R/I)\big)^2}{i\big(f,x^2\big)-\dim_{\mathbb{C}}(R/I)} = 2\cdot(k-1+\alpha)^2.$$

The in particular part then follows right away from Corollary 1. □

Since a convenient semi-quasihomogeneous power series of multiplicity 2 defines an A_k-singularity and one with a homogeneous leading form defines an ordinary multiple point, the following proposition together with the previous two gives upper bounds for all singularities defined by a convenient semi-quasihomogeneous representative.

PROPOSITION 3 (SEMIQUASIHOMOGENEOUS SINGULARITIES). *Let $\mathcal{S}_{p,q}$ be a singularity type with a convenient semi-quasihomogeneous representative $f \in R$, $q > p \geq 3$.*

Then $\gamma_\alpha^{es}(\mathcal{S}_{p,q}) \geq \frac{\left(q-(1-\alpha)\cdot\left\lfloor\frac{q}{p}\right\rfloor\right)^2}{\left\lfloor\frac{q}{p}\right\rfloor} \geq \frac{q\cdot(p-1+\alpha)^2}{p}$ *and we obtain the following upper bound for $\gamma_\alpha^{es}(f)$:*

p,q	$\gamma_\alpha^{es}(f)$
$q \geq 39$	$\leq 3\cdot(q-2+\alpha)^2$
$\frac{q}{p} \in (1,2)$	$\leq 3\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [2,4)$	$\leq 2\cdot(q-1+\alpha)^2$
$\frac{q}{p} \in [4,\infty)$	$\leq (q-1+\alpha)^2$

Proof. To see the claimed lower bound for $\gamma_\alpha^{es}(\mathcal{S}_{p,q})$ recall that (see [6])

$$I^{es}(f) = \left\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, x^\alpha y^\beta \;\middle|\; \alpha p + \beta q \geq pq \right\rangle. \tag{3}$$

In particular, $I^{es}(f) \subseteq \langle y, x^{q-\lfloor \frac{q}{p} \rfloor}\rangle$, $\dim_{\mathbb{C}}(R/I) = q - \lfloor \frac{q}{p} \rfloor$ and $i(f,y) = q$, which implies the claim.

Let now I be a complete intersection ideal with $I^{es}(f) \subseteq I$. Applying Lemma 9 and $\mathrm{d}(I) \leq q$, we first of all note that

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2 \cdot (q+1)^2}{4} \leq 2 \cdot (q-1+\alpha)^2.$$

Moreover, if $\frac{q}{p} \geq 3$, then

$$(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I) \leq \frac{(1+\alpha)^2 \cdot \left(q^2+4q+3\right)}{6} \leq (q-1+\alpha)^2.$$

since $\dim_{\mathbb{C}}(R/I) \leq \dim_{\mathbb{C}}\left(R/I^{es}(f)\right) \leq \frac{(p+1)\cdot(q+1)}{2}$ by (3).

It therefore suffices to show

$$\lambda_\alpha(f; I, g) \leq \begin{cases} 3 \cdot (q-2+\alpha)^2, & \text{if } q \geq 39, \\ 3 \cdot (q-1+\alpha)^2, & \text{if } \frac{q}{p} \in (1,2), \\ 2 \cdot (q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [2,4), \\ (q-1+\alpha)^2, & \text{if } \frac{q}{p} \in [4,\infty), \end{cases} \tag{4}$$

where $g \in I$ with $i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)$. Recall that

$$\lambda_\alpha(f; I, g) = \frac{\left(\alpha \cdot i(f,g) + (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\right)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)}.$$

Fixing I and considering $\lambda_\alpha(f; I, g)$ as a function in $i(f,g)$, where due to (11) the latter takes values between $\dim_{\mathbb{C}}(R/I)+1$ and $2\cdot\dim_{\mathbb{C}}(R/I)$, we note that the function is monotonically decreasing. In order to calculate an upper bound for $\lambda_\alpha(f; I, g)$ we may therefore replace $i(f,g)$ by some lower bound, which still exceeds $\dim_{\mathbb{C}}(R/I)+1$. Having done this we may then replace $\dim_{\mathbb{C}}(R/I)$ by an upper bound in order to find an upper bound for $\lambda(f; I, g)$.

Note that for $q \geq 39$ we have

$$\frac{54}{19} \cdot (q-1+\alpha)^2 \leq 3 \cdot (q-2+\alpha)^2. \tag{5}$$

Fix I and g, and let $L_{(p,q)}(g) = x^A y^B$ be the leading term of g w. r. t. the weighted ordering $<_{(p,q)}$ (see Definition 6). By Remark 5 we know

$$i(f,g) \geq Ap + Bq. \tag{6}$$

Working with this lower bound for $i(f,g)$ we reduce the problem to find suitable upper bounds for $\dim_{\mathbb{C}}(R/I)$. For this purpose we may assume that $L_{(p,q)}(g)$ is minimal, and thus, in particular, $B \le \text{mult}(I)$.

If $A = 0$, in view of Remark 4 we therefore have

$$B = \text{mult}(I) \le \frac{\text{d}(I)+1}{2} \le \frac{q+1}{2},$$

and thus by Lemma 9 then

$$\dim_{\mathbb{C}}(R/I) \le B \cdot (q - B + 1). \tag{7}$$

Moreover, for $A = 0$ Lemma 11 applies with $h = g$ and we get

$$\dim_{\mathbb{C}}(R/I) \le B \cdot q - 1 - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor \le B \cdot q - 1 - \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2}. \tag{8}$$

Since $x^\alpha y^\beta \in I$ for $\alpha p + \beta q \ge pq$, we may assume $Ap + Bq \le pq$. But then, since $\dim_{\mathbb{C}}(R/I) \le \dim_{\mathbb{C}} R\big/\big\langle \frac{\partial f}{\partial y}, g, x^\alpha y^\beta \mid \alpha p + \beta q \ge pq \big\rangle$, we may apply Lemma 12 with $h = \frac{\partial f}{\partial y}$ and $C = p - 1$. This gives

$$\dim_{\mathbb{C}}(R/I) \le Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \min\left\{A, \left\lceil \tfrac{q}{p} \right\rceil\right\}, \tag{9}$$

and if $B = 0$ we get in addition

$$\dim_{\mathbb{C}}(R/I) \le A \cdot (p-1). \tag{10}$$

Finally note that by Lemma 1

$$i(f,g) > \dim_{\mathbb{C}}(R/I). \tag{11}$$

Let us now use the inequalities (5)-(11) to show (4). For this we have to consider several cases for possible values of A and B.

<u>CASE 1:</u> $A = 0$, $B \ge 1$.

If $B = 1$, then by (8) and (11) we have $\lambda_\alpha(f; I, g) \le (q - 1 + \alpha)^2$.

We may thus assume that $B \ge 2$. By (6) and (7)

$$\lambda_\alpha(f; I, g) \le \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{B\cdot(B-1)} \le 2 \cdot (q - 1 + \alpha)^2.$$

If, moreover, $\frac{q}{p} \ge 3$, then we may apply (8) to find

$$\lambda_\alpha(f; I, g) \le \frac{B^2 \cdot \big(q - (1-\alpha)\cdot(B-1)\big)^2}{\left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \le (q - 1 + \alpha)^2.$$

Taking (5) into account, this proves (4) in the case $A = 0$ and $B \geq 1$.

CASE 2: $A = 1$, $B \geq 1$.

From (9) we deduce

$$\dim_{\mathbb{C}}(R/I) \leq B \cdot (q-1) + (p-1) - \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2}.$$

Since $\frac{p-1+\alpha}{q-1+\alpha} \leq \frac{p}{q}$ we thus get

$$\lambda_\alpha(f;I,g) \leq \frac{\left(B + \frac{p-1+\alpha}{q-1+\alpha}\right)^2}{B + \left\lfloor \frac{q}{p} \right\rfloor \cdot \frac{B\cdot(B-1)}{2} + 1} \cdot (q-1+\alpha)^2$$

$$\leq \begin{cases} \frac{(B+\frac{1}{3})^2}{\frac{3B^2}{2} - \frac{B}{2} + 1} \cdot (q-1+\alpha)^2 & \leq (q-1+\alpha)^2, & \text{if } \frac{q}{p} \geq 3, \\ \frac{(B+\frac{1}{2})^2}{B^2+1} \cdot (q-1+\alpha)^2 & \leq \frac{5}{4} \cdot (q-1+\alpha)^2, & \text{if } \frac{q}{p} \geq 2, \\ 2 \cdot \frac{(B+1)^2}{B^2+B+2} \cdot (q-1+\alpha)^2 & \leq \frac{16}{7} \cdot (q-1+\alpha)^2, & \text{if } \frac{q}{p} > 1. \end{cases}$$

Once more we are done, since $\frac{16}{7} \leq \frac{54}{19}$.

CASE 3: $A \geq 2$, $B \geq 1$.

Note that $\lfloor r \rfloor \geq r - 1$ for any rational number r, and set $s = \frac{q}{p}$, then by (9)

$$\dim_{\mathbb{C}}(R/I) \leq Ap + Bq - (A-1)\cdot(B-1) - \frac{A\cdot(A-1)}{2s} - \frac{s\cdot B\cdot(B-1)}{2} - 1 - \min\left\{A, \lceil s \rceil\right\}.$$

This amounts to

$$\lambda_\alpha(f;I,g) \leq$$

$$\frac{\left(Ap+Bq-(1-\alpha)\cdot\left((A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+1+\min\{A,\lceil s\rceil\}\right)\right)^2}{(A-1)\cdot(B-1)+\frac{A\cdot(A-1)}{2s}+\frac{s\cdot B\cdot(B-1)}{2}+3}$$

$$\leq \frac{\left(A\cdot(p-1+\alpha) + B\cdot(q-1+\alpha)\right)^2}{(A-1)\cdot(B-1) + \frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 3} \leq \varphi(A,B)\cdot(q-1+\alpha)^2,$$

where

$$\varphi(A,B) = \frac{\left(\frac{A}{s} + B\right)^2}{(A-1)\cdot(B-1) + \frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 3}.$$

For the last inequality we just note again that $\frac{p-1+\alpha}{q-1+\alpha} \leq \frac{p}{q} = \frac{1}{s}$, while for the second inequality a number of different cases has to be considered. We postpone this for a moment.

In order to show (4) in the case $A \geq 2$ and $B \geq 1$ it now suffices to show

$$\varphi(A,B) \leq \begin{cases} \frac{54}{19}, & \text{if } s \geq 1, \\ 2, & \text{if } s \geq 2, \\ 1, & \text{if } s \geq 4. \end{cases} \tag{12}$$

Elementary calculus shows that for $B \geq 1$ fixed the function $[2,\infty) \to \mathbb{R} : A \mapsto \varphi(A,B)$ takes its maximum at

$$A = \max\left\{2, \frac{16-3B}{2+\frac{1}{s}}\right\}.$$

If $B \leq 3$, then the maximum is attained at $A = \frac{16-3B}{2+\frac{1}{s}}$, and

$$\varphi(A,B) \leq \varphi\left(\frac{16-3B}{2+\frac{1}{s}}, B\right) = \frac{8sB-8B+64}{4s^2B-4s^2-4sB+28s-1}.$$

Again elementary calculus shows that the function $B \mapsto \varphi\left(\frac{16-3B}{2+\frac{1}{s}}, B\right)$ is monotonically decreasing on $[1,3]$ and, therefore,

$$\varphi(A,B) \leq \varphi\left(\frac{13}{2+\frac{1}{s}}, 1\right) = \frac{8s+56}{24s-1} =: \psi_1(s).$$

Since also the function ψ_1 is monotonically decreasing on $[1,\infty)$ and $\psi_1(1) = \frac{64}{23} \leq \frac{54}{19}$, $\psi_1(2) = \frac{72}{47} \leq 2$ and $\psi_1(4) = \frac{88}{95} \leq 1$ Equation (12) follows in this case.

As soon as $B \geq 4$ the maximum for $\varphi(A,B)$ is attained for $A = 2$ and

$$\varphi(A,B) \leq \varphi(2,B) = \frac{2\cdot(sB+2)^2}{s^3B^2-s^3B+2s^2B+4s^2+2s}.$$

Once more elementary calculus shows that the function $B \mapsto \varphi(2,B)$ is monotonically decreasing on $[4,\infty)$. Thus

$$\varphi(A,B) \leq \varphi(2,4) = \frac{4\cdot(1+2s)^2}{6s^3+6s^2+s} =: \psi_2(s).$$

Applying elementary calculus again, we find that the function ψ_2 is monotonically decreasing on $[1,\infty)$, so that we are done since $\psi_2(1) = \frac{36}{13} \leq \frac{54}{19}$, $\psi_2(2) = \frac{50}{37} \leq 2$ and $\psi_2(4) = \frac{81}{121} \leq 1$.

Let us now come back to proving the missing inequality above. We have to show

$$A+B \leq (A-1)\cdot(B-1) + \frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 1 + \min\left\{A, \lceil s\rceil\right\},$$

or equivalently

$$\frac{A\cdot(A-1)}{2s} + \frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\left\{A, \lceil s\rceil\right\} + AB - 2A - 2B \geq 0.$$

If $B \geq 2$, then $AB \geq 2A$ and $\frac{s\cdot B\cdot(B-1)}{2} + 2 + \min\left\{A, \lceil s\rceil\right\} \geq 2B$, so we are done. It remains to consider the case $B = 1$, and we have to show

$$A^2 - A - 2sA + 2s\cdot\min\left\{A, \lceil s\rceil\right\} \geq 0.$$

If $A \leq \lceil s \rceil$ or $A = 2$ this is obvious. We may thus suppose that $A > \lceil s \rceil$ and $A \geq 3$. Since $\frac{A^2}{3} \geq A$ it remains to show

$$\frac{2A^2}{3} - 2sA + 2s \cdot \lceil s \rceil \geq 0.$$

For this

$$\frac{2A^2}{3} - 2sA + 2s \cdot \lceil s \rceil \geq \begin{cases} \frac{2A^2}{3} - 2sA & \geq 0, \quad \text{if } A \geq 3s, \\ \frac{2A^2}{3} - \frac{4sA}{3} & \geq 0, \quad \text{if } 2s \leq A \leq 3s, \\ \frac{2A^2}{3} - sA & \geq 0, \quad \text{if } \frac{3s}{2} \leq A \leq 2s, \\ \frac{2A^2}{3} - \frac{2sA}{3} & \geq 0, \quad \text{if } \lceil s \rceil \leq A \leq \frac{3s}{2}. \end{cases}$$

<u>CASE 4:</u> $A \geq 1$, $B = 0$.

Applying (9) and (10) we get

$$\lambda_\alpha(f; I, g) \leq \begin{cases} \frac{A^2 \cdot (p-1+\alpha)^2}{A} \leq \begin{Bmatrix} \frac{A}{s^2} \cdot (q-1+\alpha)^2 \\ A \cdot (q-2+\alpha)^2 \end{Bmatrix} & \text{for any } A, \\ \frac{A^2 \cdot (p-1+\alpha)^2}{\sum_{i=1}^{A-1} \lfloor \frac{pi}{q} \rfloor + \min\{A, \lceil \frac{q}{p} \rceil\}} \leq \varphi_{\nu,s}(A) \cdot (q-1+\alpha)^2, & \text{if } A \geq 3, \end{cases}$$

where

$$\varphi_{\nu,s}(A) = \frac{\frac{A^2}{s^2}}{\frac{A \cdot (A-1)}{2s} - (A-1) + \nu} = \frac{2A^2}{sA^2 - (2s^2 + s) \cdot A + 2 \cdot (\nu + 1) \cdot s^2}$$

with $\nu = 2$ for $s \in (1, 2]$ and $\nu = 3$ for $s \in (2, \infty)$.

In particular, due to the first two inequalities we may thus assume that

$$A > \begin{cases} 3, & \text{if } q \geq 39, \\ 3s^2, & \text{if } s \in (1, 2), \\ 2s^2, & \text{if } s \in [2, 4), \\ s^2, & \text{if } s \in [4, \infty). \end{cases}$$

Note that $\varphi_{3,s}(A) \leq 1$ for $s \geq 4$, since

$$A \geq s^2 = \frac{9s^2}{16} + \frac{7s^2}{16} \geq \frac{s \cdot (1 + 2s)}{2 \cdot (s-2)} + \frac{s}{s-2} \cdot \sqrt{s^2 - 3s + \frac{33}{4}}.$$

This gives (4) for $s \geq 4$.

If now $s \in (2, 4)$, then $\varphi_{3,s}$ is monotonically decreasing on $\left[2s^2, \infty\right)$, as is $s \mapsto \varphi_{3,s}\left(2s^2\right)$ on $[2, 4)$, and thus

$$\varphi_{3,s}(A) \leq \varphi_{3,s}\left(2s^2\right) = \frac{4s^2}{2s^3 - 2s^2 - s + 4} \leq \frac{8}{5} \leq 2,$$

while for $s = 2$ the function $\varphi_{2,2}$ is monotonically decreasing on $[8, \infty)$ and thus $\varphi_{2,2}(A) \leq \frac{16}{9} \leq 2$. This finishes the case $s \in [2, 4)$.

Let's now consider the case $s \in (1, 2)$ and $q \geq 39$ parallel. Applying elementary calculus, we find that $\varphi_{2,s}$ takes its maximum on $[3, \infty)$ at $A = \frac{12s}{1+2s}$ and is monotonically decreasing on $\left[\frac{12s}{1+2s}, \infty\right)$. Moreover, the function $s \mapsto \varphi_{2,s}\left(\frac{12s}{1+2s}\right)$ is monotonically decreasing on $(1, 2)$. If $s \geq \frac{7}{6}$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,s}\left(\tfrac{12s}{1+2s}\right) \leq \varphi_{2,\frac{7}{6}}\left(\tfrac{21}{5}\right) = \frac{54}{19}.$$

Due to (5) it thus remains to consider the case $s \in \left(1, \frac{7}{6}\right)$ and $A > 3$. If $A \geq 8$, then

$$\varphi_{2,s}(A) \leq \varphi_{2,1}(8) = \frac{64}{23} \leq \frac{54}{19},$$

since the function $s \mapsto \varphi_{2,s}(8)$ is monotonically decreasing on $[1, 2)$.

So, we are finally stuck with the case $A \in \{4, 5, 6, 7\}$ and $1 \leq \frac{q}{p} = s \leq \frac{7}{6}$. We want to apply Lemma 9. For this we note first that by Lemma 13 in our situation $\mathrm{d}(I) \leq p + 1$ and $A = \mathrm{mult}(I) \leq \frac{p+2}{2}$. But then

$$\dim_{\mathbb{C}}(R/I) \leq A \cdot (p - A + 2)$$

and thus,

$$\lambda_\alpha(f; I, g) \leq \frac{A^2 \cdot \big(p - (1-\alpha) \cdot (A-2)\big)^2}{A \cdot (A-2)} \leq \frac{A}{(A-2)} \cdot (q-2+\alpha)^2 \leq 2 \cdot (q-2+\alpha)^2.$$

This finishes the proof. □

REMARK 1. In the proof of the previous proposition we achieved for almost all cases $\lambda_\alpha(f; I, g) \leq \frac{54}{19} \cdot (q - 1 + \alpha)^2$, apart from the single case $L_{<_{(p,q)}}(g) = x^3$. The following example shows that indeed in this case we cannot, in general, expect any better coefficient than 3. More precisely, the example shows that the bound

$$3 \cdot (q - 2 + \alpha)^2$$

is sharp for the family of singularities given by $x^q - y^{q-1}$, $q \geq 39$. A closer investigation should allow to lower the bound on q, but we cannot get this for all $q \geq 4$, as the example of E_6 and E_8 show.

Moreover, we give series of examples for which the bound $(q-1+\alpha)^2$ is sharp, respectively for which $2 \cdot (q - 1 + \alpha)^2$ is a lower bound.

EXAMPLE 1. Throughout these examples $q > p \geq 3$ are integers.

1. Let $f = x^q - y^{q-1}$, then $\gamma_\alpha^{es}(f) \geq 3 \cdot (q - 2 + \alpha)^2$. In particular, for $q \geq 39$,

$$\gamma_\alpha^{es}(f) = 3 \cdot (q - 2 + \alpha)^2.$$

2. Let $\frac{q}{p} < 2$ and $f = x^q - y^p$, then

$$\gamma_\alpha^{es}(f) \geq 2 \cdot (q - 1 + \alpha)^2.$$

3. Let $f \in R$ be convenient, semi-quasihomogeneous of $\operatorname{ord}_{(p,q)}(f) = pq$, and suppose that in f no monomial $x^k y$, $k \leq q-2$, occurs (e. g. $f = x^q - y^p$), then $\gamma_\alpha^{es}(f) \geq (q - 1 + \alpha)^2$. In particular, if $\frac{q}{p} \geq 4$, then

$$\gamma_\alpha^{es}(f) = (q - 1 + \alpha)^2.$$

4. Let $f = y^3 - 3x^8 y + 3x^{12}$, then f does not satisfy the assumptions of (c), but still $\gamma_\alpha^{es}(f) = (11 + \alpha)^2 = (q - 1 + \alpha)^2$.

5. Let $f = 7y^3 + 15x^7 - 21x^5 y$, then f is semi-quasihomogeneous with weights $(p, q) = (3, 7)$ and convenient, but $\gamma_0^{es}(f) \leq 25 < 36 = (q - 1)^2$. This shows that $(q - 1)^2$ is not a general lower bound for $\gamma_0^{es}(\mathcal{S}_{p,q})$.

2. Local monomial orderings

Throughout the proofs of the auxilary statements in Section 4 we make use of some results from computer algebra concerning properties of local monomial orderings. In this section we recall the relevant definitions and results.

DEFINITION 6. *A* monomial ordering *is a total ordering $<$ on the set of monomials $\{x^\alpha y^\beta \mid \alpha, \beta \geq 0\}$ such that for all $\alpha, \beta, \gamma, \delta, \mu, \nu \geq 0$*

$$x^\alpha y^\beta < x^\gamma y^\delta \quad \Longrightarrow \quad x^{\alpha+\mu} y^{\beta+\nu} < x^{\gamma+\mu} y^{\delta+\nu}.$$

A monomial ordering $<$ is called local *if $1 > x^\alpha y^\beta$ for all $(\alpha, \beta) \neq (0, 0)$, and it is a local* degree ordering *if*

$$\alpha + \beta > \gamma + \delta \quad \Longrightarrow \quad x^\alpha y^\beta < x^\gamma y^\delta.$$

Finally, if $<$ is any local monomial ordering, then we define the leading monomial *$L_<(f)$ with respect to $<$ of a non-zero power series $f \in R$ to be the maximal monomial $x^\alpha y^\beta$ such that the coefficient of $x^\alpha y^\beta$ in f does not vanish. For $f = 0$, we set $L_<(f) := 0$.*
If $I \trianglelefteq R$ is an ideal in R, then $L_<(I) = \langle L_<(f) \mid f \in I\rangle$ is called its leading ideal.

We will give now some examples of local monomial orderings which are used in the proofs.

EXAMPLE 2. Let $\alpha, \beta, \gamma, \delta \geq 0$ be integers.

1. The *negative lexicographical ordering* $<_{ls}$ is defined by the relation

$$x^\alpha y^\beta <_{ls} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha > \gamma \text{ or } (\alpha = \gamma \text{ and } \beta > \delta).$$

2. The *negative degree reverse lexicographical ordering* $<_{ds}$ is defined by the relation

$$x^\alpha y^\beta <_{ds} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \alpha+\beta > \gamma+\delta \text{ or } (\alpha+\beta=\gamma+\delta \text{ and } \beta > \delta).$$

3. If positive integers p and q are given, then we define the *local weighted degree ordering* $<_{(p,q)}$ with weights (p,q) by the relation

$$x^\alpha y^\beta <_{(p,q)} x^\gamma y^\delta \quad :\Longleftrightarrow \quad \begin{array}{l} \alpha p+\beta q > \gamma p+\delta q \text{ or} \\ (\alpha p+\beta q = \gamma p+\delta q \text{ and } \beta < \delta). \end{array}$$

We note that $<_{ds}$ is a local degree ordering, while $<_{ls}$ is not and $<_{(p,q)}$ is if and only if $p=q$.

Let us finally recall some useful properties of local orderings (see e. g. [7] Corollary 7.5.6 and Proposition 5.5.7).

PROPOSITION 4. *Let $<$ be any local monomial ordering and I a zero-dimensional ideal in R.*

1. *The monomials of $R/L_<(I)$ form a $\mathbb{C}$-basis of R/I. In particular*

$$\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big).$$

2. *If $<$ is a degree ordering, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide (see Definition 7, and see also Remark 2).*

3. The Hilbert Samuel function

A useful tool in the study of the degree of zero-dimensional schemes and their subschemes is the Hilbert Samuel function of the structure sheaf, that is of the corresponding Artinian ring.

DEFINITION 7. *Let $I \lhd R$ be a zero-dimensional ideal.*

1. *The function*

$$H^1_{R/I} : \mathbb{Z} \to \mathbb{Z} : d \mapsto \begin{cases} \dim_{\mathbb{C}}\big(R\big/(I+\mathfrak{m}^{d+1})\big), & d \geq 0,\\ 0, & d<0, \end{cases}$$

is called the Hilbert Samuel function *of R/I.*

2. *We define the* slope *of the Hilbert Samuel function of R/I to be the function*

$$H^0_{R/I} : \mathbb{N} \to \mathbb{N} : d \mapsto H^1_{R/I}(d) - H^1_{R/I}(d-1).$$

Thus

$$H^0_{R/I}(d) = \dim_{\mathbb{C}}\big(\mathfrak{m}^d\big/((I\cap\mathfrak{m}^d)+\mathfrak{m}^{d+1})\big),$$

is just the number $d+1$ of linearly independent monomials of degree d in $\mathfrak{m}^d$ minus the number of linearly independent monomials of degree d in $\left(I\cap\mathfrak{m}^d\right)+\mathfrak{m}^{d+1}$.

3. *Finally, we define the* multiplicity *of I to be*

$$\mathrm{mult}(I) := \min\left\{\mathrm{mult}(f)\;\middle|\;0\neq f\in I\right\},$$

and the degree bound *of I as*

$$\mathrm{d}(I) := \min\left\{d\in\mathbb{N}\;\middle|\;\mathfrak{m}^d\subseteq I\right\}.$$

Let us gather some straight forward properties of the slope of the Hilbert Samuel function.

LEMMA 6. *Let $J\subseteq I\lhd R$ be zero-dimensional ideals.*

1. $H^0_{R/I}(d)=d+1$ *for all* $0\leq d<\mathrm{mult}(I)$.
2. $H^0_{R/I}(d)\leq H^0_{R/I}(d-1)$ *for all* $d\geq\mathrm{mult}(I)$.
3. $H^0_{R/I}(d)\leq\mathrm{mult}(I)$.
4. $H^0_{R/I}(d)=0$ *for all* $d\geq\mathrm{d}(I)$ *and* $H^0_{R/I}\neq 0$ *for all* $d<\mathrm{d}(I)$. *In particular*

$$\dim_{\mathbb{C}}(R/I)=\sum_{d=0}^{\mathrm{d}(I)-1}H^0_{R/I}(d).$$

5. $H^0_{R/I}(d)\leq H^0_{R/J}(d)$ *for all* $d\in\mathbb{N}$.
6. $\mathrm{d}(I)$ *and* $\mathrm{mult}(I)$ *are completely determined by* $H^0_{R/I}$.

Proof. For (a) we note that $I\subseteq\mathfrak{m}^d$ for all $d\leq\mathrm{mult}(I)$ and thus

$$H^0_{R/I}(d)=\dim_{\mathbb{C}}\left(\mathfrak{m}^d/\mathfrak{m}^{d+1}\right)=d+1 \text{ for all } 0\leq d<\mathrm{mult}(I).$$

By definition we see that $H^0_{R/I}(d)$ is just the number of linearly independent monomials of degree d in $\mathfrak{m}^d$, which is $d+1$, minus the number of linearly independent monomials, say $m_1,\ldots,m_r$, of degree d in $\left(I\cap\mathfrak{m}^d\right)+\mathfrak{m}^{d+1}$. We note that then the set

$$\{xm_1,\ldots,xm_r,ym_1,\ldots,ym_r\}\subseteq\mathfrak{m}\cdot\left((I\cap\mathfrak{m}^d)+\mathfrak{m}^{d+1}\right)\subseteq\left(I\cap\mathfrak{m}^{d+1}\right)+\mathfrak{m}^{d+2}$$

contains at least $r+1$ linearly independent monomials of degree $d+1$, once r was non-zero. However, for $d=\mathrm{mult}(I)$ and $g=g_d+h.o.t\in I$ with homogeneous part $g_d\neq 0$ of degree d, we have $g_d\in\left(I\cap\mathfrak{m}^d\right)+\mathfrak{m}^{d+1}$, that is, $d=\mathrm{mult}(I)$ is the smallest integer d for which there is a monomial of degree d in $\big(I \cap \mathfrak{m}^d\big) + \mathfrak{m}^{d+1}$. Thus for $d \geq \text{mult}(I) - 1$

$$H^0_{R/I}(d+1) \leq (d+2) - (r+1) = d + 1 - r = H^0_{R/I}(d),$$

which proves (b), while (c) is an immediate consequence of (a) and (b).

If $d \geq \mathrm{d}(I)$, then $H^1_{R/I}(d) = \dim_{\mathbb{C}}(R/I)$ is independent of d, and hence we have $H^0_{R/I}(d) = 0$ for all $d \geq \mathrm{d}(I)$. In particular,

$$\sum_{i=0}^{\mathrm{d}(I)-1} H^0_{R/I}(d) = H^1_{R/I}(\mathrm{d}(I)-1) - H^1_{R/I}(-1) = \dim_{\mathbb{C}}(R/I).$$

Moreover, $\mathfrak{m}^{\mathrm{d}(I)-1} + I \neq I = I + \mathfrak{m}^{\mathrm{d}(I)}$, so that $H^0_{R/I}\big(\mathrm{d}(I)-1\big) \neq 0$, and by (b) then $H^0_{R/I}(d) \neq 0$ for all $d < \mathrm{d}(I)$. This proves (d), and (e) and (f) are obvious. □

REMARK 2. Let $<$ be a local degree ordering on R, then the Hilbert Samuel functions of R/I and of $R/L_<(I)$ coincide by Proposition 4, and hence we have as well

$$H^0_{R/I} = H^0_{R/L_<(I)}, \quad \mathrm{d}(I) = \mathrm{d}\big(L_<(I)\big), \quad \text{and} \quad \text{mult}(I) = \text{mult}\big(L_<(I)\big),$$

since by the previous lemma the multiplicity and the degree bound only depend on the slope of the Hilbert Samuel function.

REMARK 3. The slope of the Hilbert Samuel function of R/I gives rise to a histogram as the graph of the function $H^0_{R/I}$. By the Lemma 6 we know that up to $\text{mult}(I) - 1$ the histogram is just a staircase with steps of height one, and from $\text{mult}(I) - 1$ on it can only go down, which it eventually will do until it reaches the value zero for $d = \mathrm{d}(I)$. This means that we get a histogram of form shown in Figure 1. Note also, that by Lemma 6 (a) the area of the histogram is just $\dim_{\mathbb{C}}(R/I)$!

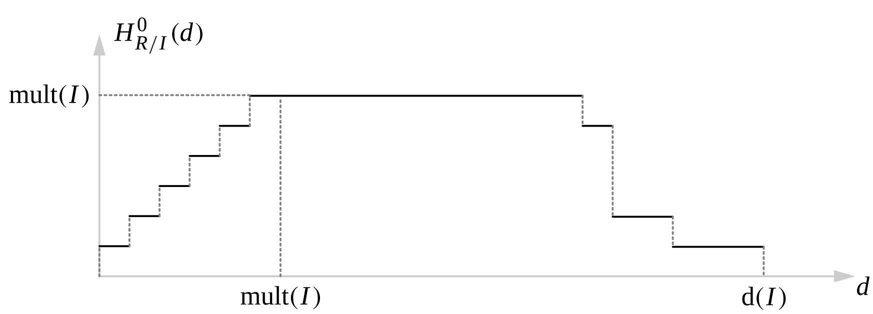

Figure 1: The histogram of $H^0_{R/I}$ for a general ideal I.

EXAMPLE 3. In order to understand the slope of the Hilbert Samuel function better, let us consider some examples.

1. Let $f = x^2 - y^{k+1}$, $k \geq 1$, and let $I = I^{ea}(f) = \langle x, y^k \rangle$ the equisingularity ideal of an A_k-singularity. Then $\mathrm{d}(I) = k$, $\mathrm{mult}(I) = 1$ and $\dim_{\mathbb{C}}(R/I) = k$.

Figure 2: The histogram of $H^0_{R/I}$ for an A_k-singularity

2. Let $f = x^2 y - y^{k-1}$, $k \geq 4$, and let $I = I^{ea}(f) = \langle xy, x^2 - (k-1) \cdot y^{k-2} \rangle$ the equisingularity ideal of a D_k-singularity. Then $x^3, xy, y^{k-1} \in I$, and thus $\mathfrak{m}^{k-1} \subset I$, which gives $\mathrm{d}(I) = k - 1$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = k$, which shows that the bound in Lemma 9 need not be obtained.

Figure 3: The histogram of $H^0_{R/I}$ for a D_k-singularity

3. Let $f = x^3 - y^4$ and let $I = I^{ea}(f) = \langle x^2, y^3 \rangle$ the equisingularity ideal of an E_6-singularity. Then $\mathrm{d}(I) = 4$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 6$.

   Let $f = x^3 - xy^3$ and let $I = I^{ea}(f) = \langle 3x^2 - y^3, xy^2 \rangle$ the equisingularity ideal of an E_7-singularity. Then $x^3, xy^2, y^5 \in I$, and thus $\mathfrak{m}^5 \subset I$, which gives $\mathrm{d}(I) = 5$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 7$.

   Let $f = x^3 - y^5$ and let $I = I^{ea}(f) = \langle x^2, y^4 \rangle$ the equisingularity ideal of an E_8-singularity. Then $\mathrm{d}(I) = 6$, $\mathrm{mult}(I) = 2$ and $\dim_{\mathbb{C}}(R/I) = 8$.

Figure 4: The histogram of $H^0_{R/I}$ for E_6, E_7 and E_8.

4. Let $I = \langle x^3, x^2 y, y^3 \rangle$, then $\mathrm{d}(I) = 4$, $\mathrm{mult}(I) = 3$ and $\dim_{\mathbb{C}}(R/I) = 7$.

The following result providing a lower bound for the minimal number of generators of a zero-dimensional ideal in R is due to A. Iarrobino.

LEMMA 7. *Let $I \lhd R$ be a zero-dimensional ideal. Then I cannot be generated by less than* $1 + \sup \left\{ H^0_{R/I}(d-1) - H^0_{R/I}(d) \;\middle|\; d \geq \mathrm{mult}(I) \right\}$ *elements.*

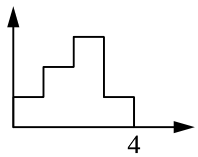

Figure 5: The histogram of $H^0_{R/I}$ for $I = \langle x^3, x^2y, y^3\rangle$.

In particular, if I is a complete intersection ideal then for $d \geq \mathrm{mult}(I)$

$$H^0_{R/I}(d-1) - 1 \leq H^0_{R/I}(d) \leq H^0_{R/I}(d-1).$$

Proof. See [8] Theorem 4.3 or [2] Proposition III.2.1. □

Moreover, by the Lemma of Nakayama and Proposition 4 we can compute the minimal number of generators for a zero-dimensional ideal exactly.

LEMMA 8. *Let $I \lhd R$ be zero-dimensional ideal and let $<$ denote any local ordering on R. Then the minimal number of generators of I is*

$$\dim_{\mathbb{C}}(I/\mathfrak{m}I) = \dim_{\mathbb{C}}\big(R/L_<(I)\big) - \dim_{\mathbb{C}}\big(R/L_<(\mathfrak{m}I)\big).$$

REMARK 4. If we apply Lemma 7 to a zero-dimensional complete intersection ideal $I \lhd R$, i. e. a zero-dimensional ideal generated by two elements, then we know that the histogram of $H^0_{R/I}$ will be as shown in Figure 6; that is, up to the value $d = \mathrm{mult}(I)$

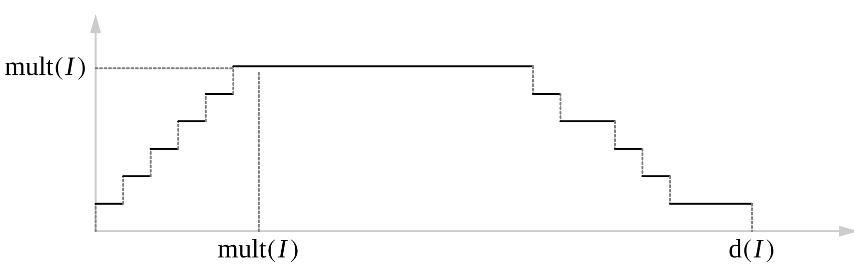

Figure 6: The histogram of $H^0_{R/I}$ for a complete intersection.

the histogram of $H^0_{R/I}$ is an ascending staircase with steps of height and length one, then it remains constant for a while, and finally it is a descending staircase again with steps of height one, but a possibly longer length. In particular we see that

$$\mathrm{mult}(I) \leq \begin{cases} \frac{\mathrm{d}(I)+1}{2}, & \text{if } \mathrm{d}(I) \text{ is odd,} \\ \frac{\mathrm{d}(I)}{2}, & \text{if } \mathrm{d}(I) \text{ is even.} \end{cases} \tag{13}$$

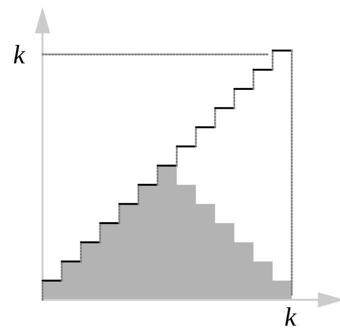

Figure 7: The histogram of $H^0_{R/\mathfrak{m}^k}$. The shaded region is the maximal possible value of $\dim_{\mathbb{C}}(R/I)$ for a complete intersection ideal I containing $\mathfrak{m}^k$.

EXAMPLE 4. Let $I = \mathfrak{m}^k$ for $k \geq 1$. Then $\mathrm{d}(I) = \mathrm{mult}(I) = k$ and $\dim_{\mathbb{C}}(R/I) = \binom{k+1}{2}$.

LEMMA 9. *Let $I \lhd R$ be a zero-dimensional complete intersection ideal, then*

$$\dim_{\mathbb{C}}(R/I) \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

In particular

$$\dim_{\mathbb{C}}(R/I) \leq \begin{cases} \frac{(\mathrm{d}(I)+1)^2}{4}, & \text{if } \mathrm{d}(I) \text{ odd}, \\ \frac{\mathrm{d}(I)^2+2\,\mathrm{d}(I)}{4}, & \text{if } \mathrm{d}(I) \text{ even}. \end{cases}$$

Proof. By Remark 3 we have to find an upper bound for the area A of the histogram of $H^0_{R/I}$. This area would be maximal, if in the descending part the steps had all length one, i. e. if the histogram was as shown in Figure 8. Since the two shaded regions have

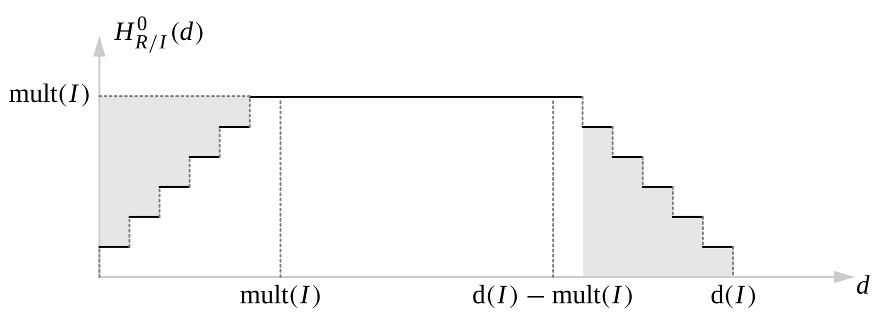

Figure 8: Maximal possible area.

the same area, we get

$$A \leq \big(\mathrm{d}(I) - \mathrm{mult}(I) + 1\big) \cdot \mathrm{mult}(I).$$

Consider now the function

$$\varphi : \left[\,\mathrm{mult}(I), \tfrac{\mathrm{d}(I)+1}{2}\right] \longrightarrow \mathbb{R} : x \mapsto \big(\mathrm{d}(I) - x + 1\big) \cdot x,$$

then this function is monotonically increasing, which finishes the proof in view of Equation (13). □

COROLLARY 1. *For an ordinary m-fold point M_m we have*

$$\tau^{es}_{ci}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \geq 3 \text{ odd},\\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even},\\ 1, & \text{if } m = 2.\end{cases}$$

Proof. Let f be a representative of M_m. Then

$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle + \mathfrak{m}^m,$$

and as in the proof of Proposition 2 we may assume that f is a homogeneous of degree m.

In particular, if $m = 2$, then $I^{es}(f) = \mathfrak{m}$ is a complete intersection and $\tau^{es}_{ci}(M_2) = 1$. We may therefore assume that $m \geq 3$.

For any complete intersection ideal I with $\mathfrak{m}^m \subset I^{es}(f) \subseteq I$ we automatically have $\mathrm{d}(I) \leq m$, and by Lemma 9

$$\tau^{es}_{ci}(f) \leq \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \text{ odd},\\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even}.\end{cases}$$

Consider now the representative $f = x^m - y^m$. If $m = 2k$ is even, then the ideal $I = \langle x^k, y^{k+1}\rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau^{es}_{ci}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 + k = \frac{m^2+2m}{4}.$$

Similarly, if $m = 2k - 1$ is odd, then the ideal $I = \langle x^k, y^k\rangle$ is a complete intersection with $I^{es}(f) \subset I$ and

$$\tau^{es}_{ci}(f) \geq \dim_{\mathbb{C}}(R/I) = k^2 = \frac{m^2+2m+1}{4}.$$

□

4. Semi-quasihomogeneous singularities

DEFINITION 8. *A non-zero polynomial of the form $f = \sum_{\alpha\cdot p+\beta\cdot q=d} a_{\alpha,\beta}x^\alpha y^\beta$ is called* quasihomogeneous *of (p, q)-degree d. Thus the Newton polygon of a quasi-homogeneous polynomial has just one side of slope $-\frac{p}{q}$.*

A quasihomogeneous polynomial is said to be non-degenerate *if it is reduced, that is if it has no multiple factors, and it is said to be* convenient *if* $\frac{d}{p}, \frac{d}{q} \in \mathbb{Z}$ *and* $a_{\frac{d}{p},0}$ *and* $a_{0,\frac{d}{q}}$ *are non-zero, that is if the Newton polygon meets the* x*-axis and the* y*-axis.*
If $f = f_0 + f_1$ *with* f_0 *quasihomogeneous of* (p, q)*-degree* d *and for any monomial* $x^\alpha y^\beta$ *occurring in* f_1 *with a non-zero coefficient we have* $\alpha \cdot p + \beta \cdot q > d$*, we say that* f *is of* (p, q)-order d*, and we call* f_0 *the* (p, q)-leading form *of* f *and denote it by* $\text{lead}_{(p,q)}(f)$*. We denote the* (p, q)*-order of* f *by* $\text{ord}_{(p,q)}(f)$*.*
A power series $f \in R$ *is said to be* semi-quasihomogeneous *with respect to the weights* (p, q) *if the* (p, q)*-leading form is non-degenerate.*

REMARK 5. Let $f \in R$ with $\deg_{(p,q)}(f) = pq$ and let f_0 denote its (p, q)-leading form.

1. If $\gcd(p, q) = r$, then f_0 has r factors of the form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$, $i = 1, \ldots, r$. If, moreover, f_0 is non-degenerate, then these will all be irreducible and pairwise different, i. e. not scalar multiples of each other.

2. If f is irreducible, then f_0 has only one irreducible factor, possibly of higher multiplicity.

3. If f_0 is non-degenerate, then f has $r = \gcd(p, q)$ branches $f_1, \ldots, f_r$, which are all semi-quasihomogeneous with irreducible (p, q)-leading form $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ for pairwise distinct points $(a_i : b_i) \in \mathbb{P}^1_{\mathbb{C}}$, $i = 1, \ldots, r$.
   The characteristic exponents of f_i are $\frac{q}{r}$ and $\frac{p}{r}$ for all $i = 1, \ldots, r$, and thus f_i admits a parametrisation of the form
$$\big(x_i(t), y_i(t)\big) = \Big(\alpha_i t^{\frac{p}{r}} + h.o.t, \beta_i t^{\frac{q}{r}} + h.o.t\Big).$$

4. If f_0 is non-degenerate, i. e. f is semi-quasihomogeneous, and $g \in R$, then
$$i(f, g) \geq \text{ord}_{(p,q)}(g).$$

Proof.

1. If $\alpha p + \beta q = pq$, then $p \mid \beta q$ and hence $p \mid \beta r$, so that $\beta \cdot \frac{r}{p}$ is a natural number. Similarly $\alpha \cdot \frac{r}{q}$ is a natural number. We may therefore consider the transformation
$$f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big) \in \mathbb{C}[x, y]_r$$
   which is a homogeneous polynomial of degree r. Thus $f_0\big(x^{\frac{r}{q}}, y^{\frac{r}{p}}\big)$ factors in r linear factors $a_i x - b_i y$, $i = 1, \ldots, r$, so that f_0 factors as
$$f_0 = \prod_{i=1}^{r} \big(a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}\big). \tag{14}$$
   Since $\gcd\left(\frac{p}{r}, \frac{q}{r}\right) = 1$, the factors $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}}$ are irreducible once neither a_i nor b_i is zero.

If f_0 is non-degenerate, then the irreducible factors of f_0 are pairwise distinct. So, $a_i = 0$ implies $r = p$ and still $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = b_i y$ irreducible, while $b_i = 0$ similarly gives $r = q$ and $a_i x^{\frac{q}{r}} - b_i y^{\frac{p}{r}} = a_i x$ irreducible. Thus, in any case the factors in (14) are irreducible and, hence, pairwise distinct.

2. With the notation from Lemma 10 and the factorisation of f_0 from (14) we get

$$g = \frac{\prod_{i=1}^{r} a_i u^{\frac{bq}{r}} v^{\frac{pq}{r^2}} - b_i u^{\frac{ap}{r}} v^{\frac{pq}{r^2}}}{u^{ap} v^{\frac{pq}{r}}} = \prod_{i=1}^{r} (a_i u - b_i).$$

By assumption f is irreducible, hence according to Lemma 10 g has at most one, possibly repeated, zero. But thus the factors of f_0 all coincide – up to scalar multiple.

3. The first assertion is an immediate consequence from (a) and (b), while the "in particular" part follows by Puiseux expansion.

4. Let g_0 be the (p, q)-leading form of g. Using the notation from (c) we have

$$\begin{aligned} i(f, g) &= \sum_{i=1}^{r} i(f_i, g) = \sum_{i=1}^{r} \operatorname{ord}\big(g(x_i(t), y_i(t))\big) \\ &= \sum_{i=1}^{r} \operatorname{ord}\Big(g_0\big(\alpha_i t^{\frac{p}{r}}, \beta_i t^{\frac{q}{r}}\big) + h.o.t\Big) \geq \sum_{i=1}^{r} \frac{\operatorname{ord}_{(p,q)}(g)}{r} = \operatorname{ord}_{(p,q)}(g). \end{aligned}$$

□

LEMMA 10. *Let* $f \in R$ *with* $\operatorname{ord}_{(p,q)}(f) = pq$ *and let* f_0 *denote its* (p, q)-*leading form. Let* $r = \gcd(p, q)$ *and* $a, b \geq 0$ *such that* $qb - pa = r$. *Finally set*

$$g = \frac{f_0\big(u^b v^{\frac{p}{r}}, u^a v^{\frac{q}{r}}\big)}{u^{ap} v^{\frac{pq}{r}}} \in \mathbb{C}[u].$$

Then the number of different zeros of g *is a lower bound for the number of branches of* f.

Proof. See [1] Remark on p. 480. □

The following investigations are crucial for the proof of Proposition 3.

LEMMA 11. *Let* $f \in R$ *be convenient semi-quasihomogeneous with leading form* f_0 *and* $\operatorname{ord}_{(p,q)}(f) = pq$, *let* $I = \big\langle x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle$, *and let* $h \in R$. *Then*

$$\dim_{\mathbb{C}} R/\big(\langle h\rangle + I^{es}(f)\big) < \dim_{\mathbb{C}} R/\big(\langle h\rangle + I\big).$$

In particular, if $L_{(p,q)}(h) = y^B$ *with* $B \leq p$, *then*

$$\dim_{\mathbb{C}} R/\big(\langle h\rangle + I^{es}(f)\big) \leq Bq - 1 - \sum_{i=1}^{B-1} \left\lfloor \frac{qi}{p} \right\rfloor.$$

Proof. As
$$I^{es}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle + I,$$
it suffices to show that
$$I^{es}(f) \not\subseteq \langle h \rangle + I,$$
which is the same as showing that not both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ belong to $\langle h \rangle + I$.

Suppose the contrary, that is, there are $h_x, h_y \in R$ such that
$$\frac{\partial f}{\partial x} \equiv h_x \cdot h \pmod{I} \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv h_y \cdot h \pmod{I}.$$
We note that
$$\mathrm{lead}_{(p,q)}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f_0}{\partial x} \quad \text{and} \quad \mathrm{lead}_{(p,q)}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f_0}{\partial y},$$
and none of the monomials involved is contained in I. Therefore
$$\mathrm{lead}_{(p,q)}(h_x) \cdot \mathrm{lead}_{(p,q)}(h) = \frac{\partial f_0}{\partial x} \quad \text{and} \quad \mathrm{lead}_{(p,q)}(h_y) \cdot \mathrm{lead}_{(p,q)}(h) = \frac{\partial f_0}{\partial y},$$
which in particular implies that $\frac{\partial f_0}{\partial x}$ and $\frac{\partial f_0}{\partial y}$ have a common factor. This, however, is then a multiple factor of the quasihomogeneous polynomial f_0, in contradiction to f being semi-quasihomogeneous.

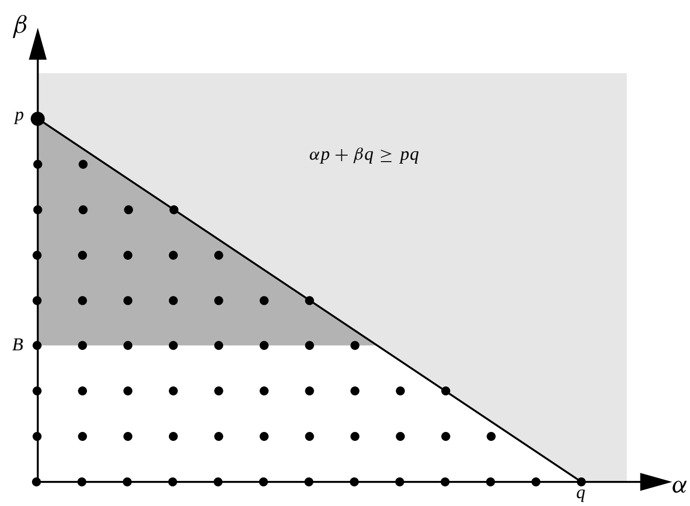

Figure 9: A Basis of $R/\langle h \rangle + I$.

For the "in particular" part, we note that by Proposition 4
$$\dim_{\mathbb{C}} R/\big(\langle h \rangle + I\big) = \dim_{\mathbb{C}} R/L_{<_{(p,q)}}\big(\langle h \rangle + I\big) \leq \dim_{\mathbb{C}} R/\big(\langle y^B \rangle + I\big),$$

and the monomials $x^\alpha y^\beta$ with $\alpha p + \beta q < pq$ and $\beta < B$ form a $\mathbb{C}$-basis of the latter vector space (see also Figure 9). Hence,

$$\dim_{\mathbb{C}} R/\big(\langle h\rangle + I\big) \leq \sum_{i=0}^{B-1} \left\lceil q - \tfrac{qi}{p} \right\rceil = Bq - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor.$$

□

LEMMA 12. *Let $g, h \in R$ such that $L_{(p,q)}(g) = x^A y^B$ and $L_{(p,q)}(h) = y^C$, and consider the ideals $J = \big\langle x^A y^B, y^C, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle$ and $J' = \big\langle g, h, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle$. Then*

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/J,$$

and if $Ap + Bq \leq pq$ and $B \leq C \leq p$, then

$$\dim_{\mathbb{C}} R/J = Ap + Bq - AB - \sum_{i=1}^{A-1} \left\lfloor \tfrac{pi}{q} \right\rfloor - \sum_{i=1}^{B-1} \left\lfloor \tfrac{qi}{p} \right\rfloor - \sum_{i=C}^{p-1} \min \left\{ A, \left\lceil q - \tfrac{Cq}{p} \right\rceil \right\}.$$

Moreover, if $B = 0$, then $\dim_{\mathbb{C}} R/J \leq A \cdot C$.

Proof. By Proposition 4

$$\dim_{\mathbb{C}} R/J' \leq \dim_{\mathbb{C}} R/L_{<_{(p,q)}}(J') \leq \dim_{\mathbb{C}} R/J.$$

Let $I = \big\langle x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle$. Then the monomials $x^\alpha y^\beta$ with $(\alpha, \beta) \in \Lambda = \big\{ (\alpha, \beta) \in \mathbb{N} \times \mathbb{N} \;\big|\; \alpha p + \beta q < pq \big\}$ form a basis of R/I. Moreover, the monomials $x^\alpha y^\beta$ with $(\alpha, \beta) \in \Lambda_1 \cup \Lambda_2$ are a basis of J/I, where

$$\Lambda_1 = \big\{ (\alpha, \beta) \in \Lambda \;\big|\; \alpha \geq A \text{ and } \beta \geq B \big\}$$

and

$$\Lambda_2 = \big\{ (\alpha, \beta) \in \Lambda \setminus \Lambda_1 \;\big|\; \beta \geq C \big\}.$$

(See also Figure 10.) This gives rise to the above values for $\dim_{\mathbb{C}} R/J$. □

LEMMA 13. *Let $q > p$ be such that $\frac{q}{p} < \frac{d}{d-1}$ for some integer $d \geq 2$, and let $0 \leq A \leq d$.*

1. *If $L_{(p,q)}(g) = x^A$, then $L_{<_{ds}}(g) = x^A$.*
2. $\mathfrak{m}^{p+1} \subseteq \big\langle x^A, y^{p-1}, x^\alpha y^\beta \;\big|\; \alpha p + \beta q \geq pq \big\rangle$.
3. *If I is an ideal such that $g, h, x^\alpha y^\beta \in I$ for $\alpha p + \beta q \geq pq$ and where $L_{<_{(p,q)}}(g) = x^A$ and $L_{<_{(p,q)}}(h) = y^{p-1}$, then $\mathrm{d}(I) \leq p + 1$.*
   Moreover, if $L_{<_{(p,q)}}(g)$ is minimal among the leading monomials of elements in I w. r. t. $<_{(p,q)}$, then $\mathrm{mult}(I) = A$.

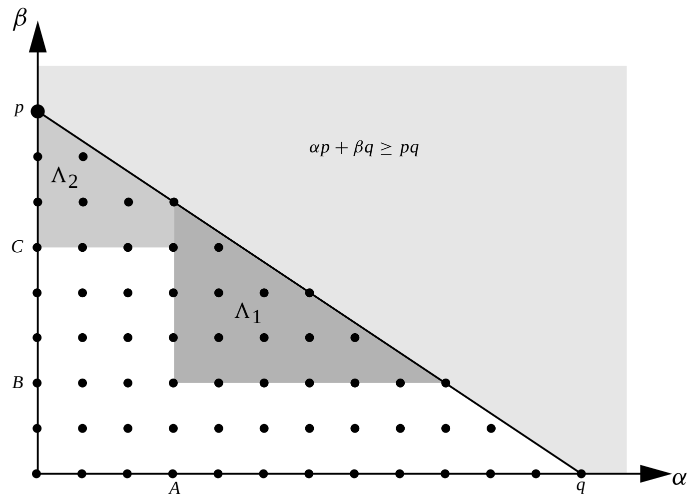

Figure 10: A Basis of R/J.

Proof. It suffices to consider the case $A = d$, since this implies the other cases. Note that by assumption $d \leq p$.

1. Since x^d is less than any monomial of degree at least d with respect to $<_{ds}$, we have to show that in g no monomial of degree less than d can occur with a non-zero coefficient. x^d being the leading monomial of g with respect to $<_{(p,q)}$, it suffices to show that $\alpha + \beta < d$ implies $\alpha p + \beta q < dp$, or alternatively, since $\frac{q}{p} < \frac{d}{d-1}$,
$$\alpha + \beta \cdot \frac{d}{d-1} \leq d.$$
For $\alpha + \beta < d$ the left hand side of this inequality will be maximal for $\alpha = 0$ and $\beta = d - 1$, and thus the inequality is satisfied.

2. We only have to show that $x^\gamma y^{p+1-\gamma} \in \left\langle x^d, y^{p-1}, x^\alpha y^\beta \;\middle|\; \alpha p + \beta q \geq pq \right\rangle$ for $\gamma = 3, \ldots, d-1$, since the remaining generators of $\mathfrak{m}^{p+1}$ definitely are. However, by assumption $\frac{q}{p} < \frac{d}{d-1} \leq \frac{\gamma}{\gamma-1}$, and thus $\gamma \cdot p + (p+1-\gamma) \cdot q \geq pq$.

3. By the assumption on I we deduce form (a) and (b) that $\mathrm{d}\left(L_{<_{ds}}(I)\right) \leq p+1$. However, by Remark 2 $\mathrm{d}(I) = \mathrm{d}\left(L_{<_{ds}}(I)\right)$, which proves the first assertion.

   Suppose now that $\mathrm{mult}(I) < A$, i. e. there is an $f \in I$ such that $\mathrm{mult}(f) \leq A-1$. The considerations for (a) show that then $L_{<_{(p,q)}}(f) < x^A$ in contradiction to the assumption.

□

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QUASI-HOMOGENEOUS LINEAR SYSTEMS ON $\mathbb{P}^2$ WITH BASE POINTS OF MULTIPLICITY 6

Abstract. In this paper we prove the Harbourne-Hirschowitz conjecture for quasi-homogeneous linear systems of multiplicity 6 on $\mathbb{P}^2$. For the proof we use the degeneration of the plane by Ciliberto and Miranda and results by Laface, Seibert, Ugaglia and Yang. As an application we derive a classification of the special systems of multiplicity 6.

1. Introduction

A classical problem in algebraic geometry is the dimensionality problem for plane curves, which can be formulated as follows. Given finitely many general points of the projective plane with assigned multiplicities and a number d, determine the dimension of the linear system of curves of degree d having at the given points at least the assigned multiplicities. More precisely, the problem is to classify all systems which fail to have the expected dimension (see [1] for some remarks on the history of this problem and its geometric meaning). Harbourne and Hirschowitz conjecture that these special systems are precisely the (-1)-special systems. In this paper, we give a complete list of the (-1)-special systems in the case in which the assigned multiplicity is 6 at all but one of the given points. Our main result is the proof of the Harbourne-Hirschowitz conjecture in this case.

We proceed along the following lines. In Section 2 we introduce the necessary notation and give a precise statement of the Harbourne-Hirschowitz conjecture. In Section 3 we present a list of the (-1)-special linear systems in our case. Its completeness is proved in Section 4. In Section 5 we review the degeneration of the plane by Ciliberto and Miranda. This method is the key tool in our proof of the main result which is given in the final two sections.

2. The Harbourne-Hirschowitz conjecture

We work over the complex numbers and choose $n+1$ general points $p_0, p_1, \dots, p_n$ in $\mathbb{P}^2$, the projective plane over that field.

NOTATION 1. We write $\mathcal{L} = \mathcal{L}(d, m_0, m_1, \dots, m_n) \subset \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$ for the linear system of all curves of degree d in $\mathbb{P}^2$ having multiplicity at least m_i at p_i for all i. We denote by $\ell(\mathcal{L})$ its projective dimension.

Let $\mathbb{P}'$ be the blow-up of $\mathbb{P}^2$ at $p_0, p_1, \dots, p_n$. By H we denote the pull-back of a line in $\mathbb{P}^2$ and by E_i the exceptional divisor over p_i. The dimension of $\mathcal{L}$ is the same as the dimension of $|D|$ on $\mathbb{P}'$ with $D = dH - m_0E_0 - m_1E_1 - \dots - m_nE_n$. Using cohomology on $\mathbb{P}'$, we have

$$\ell(\mathcal{L}) = h^0(\mathcal{O}_{\mathbb{P}'}(D)) - 1.$$

Therefore we have by Riemann-Roch

$$\ell(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2} + h^1(\mathcal{O}_{\mathbb{P}'}(D)) - h^2(\mathcal{O}_{\mathbb{P}'}(D)) + \chi(\mathcal{O}_{\mathbb{P}'}) - 1$$

($K_{\mathbb{P}'}$ denotes the canonical divisor on $\mathbb{P}'$). Since the arithmetic genus of $\mathbb{P}'$ is zero, Serre duality implies

$$\ell(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2} + h^1(\mathcal{O}_{\mathbb{P}'}(D)).$$

DEFINITION 1. *We define the virtual dimension $v(\mathcal{L})$ of $\mathcal{L}$ as follows:*

$$v(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2}.$$

We define the expected dimension to be

$$e(\mathcal{L}) = \max\{-1, v(\mathcal{L})\}.$$

As $v(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=0}^{n} \frac{m_i(m_i+1)}{2}$, one sees that the expected dimension is the one we obtain if all conditions imposed on the base points are independent.

We define $\mathcal{L}$ to be special or non-regular if

$$\ell(\mathcal{L}) > e(\mathcal{L}),$$

otherwise we call $\mathcal{L}$ non-special or regular.

We recall some definitions from [2]:

DEFINITION 2 ((-1)-SPECIAL SYSTEMS). *Let $\mathcal{A}$ in $\mathbb{P}^2$ be an irreducible curve such that its strict transform $\tilde{\mathcal{A}}$ in $\mathbb{P}'$ is rational and smooth. Then $\mathcal{A}$ is a (-1)-curve if the self-intersection number*

$$\tilde{\mathcal{A}}^2 = -1.$$

By $\mathcal{L}.\mathcal{A}$ we denote the intersection number $D.\tilde{\mathcal{A}}$ on $\mathbb{P}'$.

The linear system $\mathcal{L}$ is called (-1)-special if

- *there exist $\mathcal{A}_1, \ldots, \mathcal{A}_t$ (-1)-curves with $\mathcal{L}.\mathcal{A}_i = -n_i$ such that $n_i \geq 1$ for all i,*
- *there is an j with $n_j \geq 2$ and*
- *the residual system $\mathcal{M} = \mathcal{L} - \sum_{i=0}^{t} n_i \mathcal{A}_i$ has $v(\mathcal{M}) \geq 0$.*

The main conjecture can be formulated as follows:

CONJECTURE 1 (HARBOURNE-HIRSCHOWITZ). A linear system $\mathcal{L} = \mathcal{L}(d\ m_0, m_1, \ldots, m_n)$ is special if and only if it is (-1)-special.

It is easy to see that a (-1)-special system $\mathcal{L}$ is special because

$$v(\mathcal{L}) = \frac{\mathcal{L}.(\mathcal{L} - K_{\mathbb{P}'})}{2} = \frac{(\mathcal{M} + n\mathcal{A}).(\mathcal{M} + n\mathcal{A} - K_{\mathbb{P}'})}{2}.$$

Since $\mathcal{A}.K_{\mathbb{P}'} = -1$ by the rationality of $\tilde{\mathcal{A}}$, this implies

$$v(\mathcal{L}) = v(\mathcal{M}) + \frac{-n^2 + n}{2} \leq \ell(\mathcal{L}) + \frac{-n^2 + n}{2}.$$

Therefore the opposite direction of the Harbourne-Hirschowitz conjecture is the non-trivial one. It states that every special system $\mathcal{L}$ has fixed multiple (-1)-curves. Proving the conjecture leads to an answer of the dimensionality problem.

REMARK 1. We give a list of results on the conjecture. In fact we use all of them in several ways for the proof of our main theorem.

We write $\mathcal{L} = \mathcal{L}(d, m_0^{b_0}, m_1^{b_1}, \ldots, m_r^{b_r})$ if $\mathcal{L}$ has precisely b_i base points of multiplicity m_i for $i = 0, \ldots, r$. With this notation the conjecture holds if

- $b_0 + \ldots + b_r \leq 9$ [5],
- $\mathcal{L} = \mathcal{L}(d, m^n)$ (call it *homogeneous of multiplicity* m) and $m \leq 12$ [3],
- $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ (call it *quasi-homogeneous of multiplicity* m) and $m \leq 3$ [2],
- $\mathcal{L} = \mathcal{L}(d, m_0, 4^n)$ [9] and [7],
- $\mathcal{L} = \mathcal{L}(d, m_0, 5^n)$ [8] or
- all multiplicities are bounded by 6, i.e. $m_i \leq 6$ for $i = 0, 1, \ldots, n$ [10].

3. Main results

Our main result is a proof of the Harbourne-Hirschowitz conjecture in the quasi-homogeneous case of multiplicity 6:

THEOREM 1 (MAIN THEOREM). *A system $\mathcal{L}(d, m_0, 6^n)$ is special if and only if it is (-1)-special.*

We give the proof within an extra section. For the proof we need the following classification:

THEOREM 2 (CLASSIFICATION OF (-1)-SPECIAL SYSTEMS $\mathcal{L}(d, m_0, 6^n)$). *The following is a complete list of all (-1)-special systems $\mathcal{L}(d, m_0, 6^n)$.*

$d - m_0$	*system*	$v(\mathcal{L})$	$\ell(\mathcal{L})$	
0	$\mathcal{L}(d, d, 6^n)$	$-21n + d$	$-6n + d$	$d \geq 6n \geq 6$
1	$\mathcal{L}(d, d-1, 6^n)$	$-21n + 2d$	$-11n + 2d$	$d \geq \frac{11}{2}n \geq \frac{11}{2}$
2	$\mathcal{L}(10e, 10e-2, 6^{2e})$	$-12e - 1$	0	$e \geq 1$
	$\mathcal{L}(d, d-2, 6^n)$	$-21n + 3d - 1$	$-15n + 3d - 1$	$d \geq \frac{1+15n}{3} \geq \frac{16}{3}$
3	$\mathcal{L}(9e, 9e-3, 6^{2e})$	$-6e - 3$	0	$e \geq 1$
	$\mathcal{L}(9e+1, 9e-2, 6^{2e})$	$-6e + 1$	2	$e \geq 1$
	$\mathcal{L}(d, d-3, 6^n)$	$-21n + 4d - 3$	$\geq -18n + 4d - 3$ $=$ *if* $d \neq \frac{9n}{2} + 1$ *or* n *odd*	$d \geq \frac{18n+3}{4} \geq \frac{21}{4}$
4	$\mathcal{L}(8e, 8e-4, 6^{2e})$	$-2e - 6$	0	$e \geq 1$
	$\mathcal{L}(8e+1, 8e-3, 6^{2e})$	$-2e - 1$	2	$e \geq 1$
	$\mathcal{L}(8e+2, 8e-2, 6^{2e})$	$-2e + 4$	5	$e \geq 1$
	$\mathcal{L}(d, d-4, 6^n)$	$-21n + 5d - 6$	$\geq -20n + 5d - 6$ $=$ *if* $d \neq 4n + 2$ *or* n *odd*	$d \geq \frac{20n+6}{5} \geq \frac{26}{5}$
5	$\mathcal{L}(7e, 7e-5, 6^{2e})$	-10	0	$e \geq 1$
	$\mathcal{L}(7e+1, 7e-4, 6^{2e})$	-4	2	$e \geq 1$
	$\mathcal{L}(7e+2, 7e-3, 6^{2e})$	2	5	$e \geq 1$
	$\mathcal{L}(7e+3, 7e-2, 6^{2e})$	8	9	$e \geq 1$
6	$\mathcal{L}(6e, 6e-6, 6^{2e})$	-15	0	$e \geq 1$
	$\mathcal{L}(6e+1, 6e-5, 6^{2e})$	-8	2	$e \geq 1$
	$\mathcal{L}(6e+2, 6e-4, 6^{2e})$	-1	5	$e \geq 1$
	$\mathcal{L}(6e+3, 6e-3, 6^{2e})$	6	9	$e \geq 1$
	$\mathcal{L}(6e+4, 6e-2, 6^{2e})$	13	14	$e \geq 1$
7	$\mathcal{L}(5e+2, 5e-5, 6^{2e})$	$-2e - 5$	$-2e + 5$	$2 \geq e \geq 1$
	$\mathcal{L}(5e+3, 5e-4, 6^{2e})$	$-2e + 3$	$-2e + 9$	$4 \geq e \geq 1$
	$\mathcal{L}(5e+4, 5e-3, 6^{2e})$	$-2e + 11$	$-2e + 14$	$7 \geq e \geq 1$
	$\mathcal{L}(5e+5, 5e-2, 6^{2e})$	$-2e + 19$	$-2e + 20$	$10 \geq e \geq 1$
8	$\mathcal{L}(4e+4, 4e-4, 6^{2e})$	$-6e + 8$	$-6e + 14$	$2 \geq e \geq 1$
	$\mathcal{L}(4e+5, 4e-3, 6^{2e})$	$-6e + 17$	$-6e + 20$	$2 \geq e \geq 1$
	$\mathcal{L}(4e+6, 4e-2, 6^{2e})$	$-6e + 26$	$-6e + 27$	$4 \geq e \geq 1$
	$\mathcal{L}(10, 2, 6^3)$	-1	2	
	$\mathcal{L}(24, 16, 6^9)$	-1	0	
9	$\mathcal{L}(3e+6, 3e-3, 6^{2e})$	$-12e + 24$	$-12e + 27$	$2 \geq e \geq 1$
	$\mathcal{L}(3e+7, 3e-2, 6^{2e})$	$-12e + 34$	$-12e + 35$	$2 \geq e \geq 1$
	$\mathcal{L}(9, 0, 6^3)$	-9	0	
	$\mathcal{L}(10, 1, 6^3)$	1	4	
	$\mathcal{L}(14, 5, 6^5)$	-1	0	
	$\mathcal{L}(18, 9, 6^7)$	-3	0	
10	$\mathcal{L}(2e+8, 2e-2, 6^{2e})$	$-20e + 43$	$-20e + 44$	$2 \geq e \geq 1$
	$\mathcal{L}(10, 0, 6^3)$	2	5	

$d - m_0$	*system*	$v(\mathcal{L})$	$\ell(\mathcal{L})$
	$\mathcal{L}(14, 4, 6^5)$	4	5
11	$\mathcal{L}(13, 2, 6^5)$	-4	2
	$\mathcal{L}(14, 3, 6^5)$	8	9
12	$\mathcal{L}(12, 0, 6^5)$	-15	0
	$\mathcal{L}(13, 1, 6^5)$	-2	4
	$\mathcal{L}(14, 2, 6^5)$	11	12
13	$\mathcal{L}(13, 0, 6^5)$	-1	5
	$\mathcal{L}(14, 1, 6^5)$	13	14
14	$\mathcal{L}(14, 0, 6^5)$	14	15

4. The classification

In the paper [2] of Ciliberto and Miranda a lot of classification work has been done which we can apply to our problem. Ciliberto and Miranda introduced two notions which we recall now to use their results.

Let $\mathcal{L}$ be a linear system of plane curves with general multiple base points as above. Then $\mathcal{L}$ is a *quasi-homogeneous (-1)-class* if $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$, on $\mathbb{P}'$ the self-intersection number $\mathcal{L}.\mathcal{L} = -1$ and the arithmetic genus

$$g_{\mathcal{L}} = \frac{\mathcal{L}^2 + \mathcal{L}.K_{\mathbb{P}'}}{2} + 1 = 0.$$

As $v(\mathcal{L}) = \mathcal{L}^2 - g_{\mathcal{L}} + 1$, these systems are never empty.

In this case, if $\mathcal{A}$ is a (-1)-curve such that $\mathcal{A} \in \mathcal{L}$ then by $\mathcal{L}.\mathcal{A} = -1$ and the irreducibility of $\mathcal{A}$, we have $\mathcal{L} = \{\mathcal{A}\}$. So we can identify (-1)-curves and quasi-homogeneous (-1)-classes and write $\mathcal{A} = \mathcal{L}$. Ciliberto and Miranda proved that such a (-1)-curve exists up to $m \leq 6$. Hence a numerical classification of these systems gives a classification for all quasi-homogeneous (-1)-curves up to multiplicity $m = 6$. Such a classification is given in [2].

Now we consider the following phenomenon: Let $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ be a quasi-homogeneous linear system and $\mathcal{A}$ a (-1)-curve such that $\mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_1, \ldots, \mu_n)$ and $\mathcal{L}.\mathcal{A} \leq -2$. Let Perm_n be the permutation group on n letters and let $\sigma \in \mathrm{Perm}_n$. We define $\mathcal{A}_\sigma = \mathcal{L}(\delta, \mu_0, \mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)})$. Then, as $\mathcal{A}$ is a (-1)-curve, it follows that $\mathcal{A}_\sigma$ is again a (-1)-curve. As $\mathcal{L}$ is quasi-homogeneous we have again $\mathcal{L}.\mathcal{A}_\sigma \leq -2$. Therefore we can construct a composition of (-1)-curves, which split off the system $\mathcal{L}$. We define the set $A \subset \mathrm{Perm}_n$ to be maximal such that all $\mathcal{A}_\sigma$ with $\sigma \in A$ are pairwise different. Then we define a new plane curve $\mathcal{A}_{tot} = \sum_{\sigma \in A} \mathcal{A}_\sigma$ (see [8]).

We call a linear system $\mathcal{L}' = \mathcal{L}(d, m_o, m_1, \ldots, m_n)$ as above a *quasi-homogeneous (-1)-configuration* if $\mathcal{A}_{\mathrm{tot}}$ is a generic element in $\mathcal{L}'$. We note that $\mathcal{L}'$ is by construction quasi-homogeneous (if $k = |A|$ then there exists a μ' such that $\mathcal{L}' = \mathcal{L}(k\delta, k\mu_0, \mu'^n)$).

LEMMA 1 (SPLITTING-OFF LEMMA). *Let $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$. Then every (-1)-curve $\mathcal{A}$ with $\mathcal{L}.\mathcal{A} \leq -2$ is of one of the following types (We have listed the associated quasi-homogeneous compound (-1)-configurations, too.):*

$$\mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_1^n)$$
$$\mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_2 - 1, \mu_2^{n-1}) \quad \mathcal{A}_{tot} = \mathcal{L}(n\delta, n\mu_0, (n\mu_2 - 1)^n)$$
$$\mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_2 + 1, \mu_2^{n-1}) \quad \mathcal{A}_{tot} = \mathcal{L}(n\delta, n\mu_0, (n\mu_2 + 1)^n)$$

Proof. First one proves that strict transforms of different $\mathcal{A}_\sigma \neq \mathcal{A}_{\sigma'}$ cannot meet positively on $\mathbb{P}'$. This is the case as otherwise one sees, by the Riemann-Roch theorem on $\mathbb{P}'$, that the sum of these moves in a linear system of positive dimension, which is a contradiction to being a fixed part of $\mathcal{L}$. This implies that all the different $\mathcal{A}_\sigma$ are linearly independent in $\mathrm{Pic}(\mathbb{P}')$. Let the $\mu_1, \ldots, \mu_n$ occur in sets of size $k_1 \leq \ldots \leq k_s$. As rank $\mathrm{Pic}(\mathbb{P}') = n + 2$ we see by combinatorial reasons that for the $\frac{n!}{k_1! \cdots k_s!}$ different (-1)-curves $\mathcal{A}_\sigma$ only the possibilities

$$s = 1,\ k_1 = n \qquad \text{or}$$
$$s = 2,\ k_1 = 1,\ k_2 = n - 1$$

can occur. That means we have at most three different multiplicities μ_0, μ_1 and μ_2.

Moreover we have the equations $\mathcal{A}.\mathcal{A} = -1$ and $\mathcal{A}.\mathcal{A}_\sigma = 0$ on $\mathbb{P}'$. That gives $\mathcal{A}.\mathcal{A} - \mathcal{A}.\mathcal{A}_\sigma = -1$ which is equivalent to $(\mu_1 - \mu_2)^2 = 1$ (see [2]). □

For the purpose of classifying the systems $\mathcal{L}(d, m_0, 6^n)$ we need a complete list of all (-1)-curves which might split off such systems two times. These (-1)-curves can not have higher multiplicities than 3 at the points $p_1, \ldots, p_n$. We obtain the following result:

LEMMA 2 (CLASSIFICATION OF (-1)-CURVES). *All (-1)-curves $\mathcal{A}$ and quasi-homogeneous (-1)-configurations $\mathcal{A}_{\text{tot}}$ up to multiplicity 3 in the points $p_1, \ldots, p_n$ which might split off a quasi-homogeneous system $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ are elements of the systems in the following list (see [8]):*

not compound	*compound*
$\mathcal{L}(2, 0, 1^5)$	
$\mathcal{L}(e, e-1, 1^{2e})\quad e \geq 1$	
$\mathcal{L}(1, 1, 1^1)$	$\mathcal{L}(n, n, 1^n)\quad n \geq 2$
$\mathcal{L}(1, 0, 1^2)$	$\mathcal{L}(3, 0, 2^3)$
$\mathcal{L}(6, 3, 2^7)$	
$\mathcal{L}(12, 8, 3^9)$	

In particular, all the (-1)-curves are quasi-homogeneous.

Proof. We refer to [2, Example 5.1] for the proof of a list of all quasi-homogene- ous (-1)-classes up to multiplicity 3. In [2, Example 5.15] is given a complete list of all quasi-homogeneous (-1)-configurations up to multiplicity 3. Using this two lists and Lemma 1 gives this result. □

Now we give the proof of the classification theorem of all (-1)-special systems of the form $\mathcal{L}(d, m_0, 6^n)$.

Proof of Theorem 2. In lemma 2 we have seen the possible cases for (-1)-curves which might split off $\mathcal{L}(d, m_0, 6^n)$. Now we have to consider all these cases. To be a little bit faster we proceed along the following algorithm (see [8]):

We go through all possible combinations of these (-1)-curves step by step.

First step: If we find a (-1)-curve or a (-1)-configuration $\mathcal{A}$ such that

$$\mathcal{L}.\mathcal{A} = -\mu \leq -2,$$

then we split off the fixed part and define $\mathcal{M} = \mathcal{L} - \mu \cdot \mathcal{A}$.

Second step: Let $\mathcal{M}'$ be the residual system of $\mathcal{M}$ obtained by splitting off all possible (-1)-curves. By the definition of (-1)-special systems we have to verify that $v(\mathcal{M}') \geq 0$. We notice that the systems $\mathcal{M}$ are quasi-homogeneous of multiplicity ≤ 4 by Lemma 2. Therefore we can use the results of [2] and [9].

We give an impression of this procedure. The complete proof can be found in the extended version of this paper (cf. [6]):

- $\boxed{\mathcal{L} = \mathcal{M} + \mu \cdot \mathcal{A}, \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

1. $\mathcal{A} = \mathcal{L}(\mathbf{2}, \mathbf{0}, \mathbf{1^5})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^5})$
This gives $\mathcal{M} = \mathcal{L}(d-2n, m_0, (6-\mu)^5)$ and $\mathcal{M}.\mathcal{A} = 0$ gives $d = \frac{30-\mu}{2}$.

If $\underline{\mu = 2} \Longrightarrow d = 14$ and we get

$m_0 = 0$ and $v(\mathcal{M}) = 15$ with $\mathcal{M} = \mathcal{L}(10, 0, 4^5)$ which is non-special by [9]
$m_0 = 1$ and $v(\mathcal{M}) = 14$ with $\mathcal{M} = \mathcal{L}(10, 1, 4^5)$ ″
$m_0 = 2$ and $v(\mathcal{M}) = 12$ with $\mathcal{M} = \mathcal{L}(10, 2, 4^5)$ ″
$m_0 = 3$ and $v(\mathcal{M}) = 9$ with $\mathcal{M} = \mathcal{L}(10, 3, 4^5)$ ″
$m_0 = 4$ and $v(\mathcal{M}) = 5$ with $\mathcal{M} = \mathcal{L}(10, 4, 4^5)$ ″
$m_0 = 5$ and $v(\mathcal{M}) = 0$ with $\mathcal{M} = \mathcal{L}(10, 5, 4^5)$ ″

$\underline{\mu = 3}$ is not possible because of $\mathcal{M}.\mathcal{A} = 0$.

If $\underline{\mu = 4} \Longrightarrow d = 13$ and we conclude

$m_0 = 0$ and $v(\mathcal{M}) = 5$ with $\mathcal{M} = \mathcal{L}(7, 0, 2^5)$ which is non-special by [2]
$m_0 = 1$ and $v(\mathcal{M}) = 4$ with $\mathcal{M} = \mathcal{L}(7, 1, 2^5)$ ″
$m_0 = 2$ and $v(\mathcal{M}) = 2$ with $\mathcal{M} = \mathcal{L}(7, 2, 2^5)$ ″
$m_0 = 3$ and $v(\mathcal{M}) = -1$

$\underline{\mu = 5}$ is not possible because of $\mathcal{M}.\mathcal{A} = 0$.

From $\underline{\mu = 6} \Longrightarrow d = 12$ and $m_0 = 0$, $v(\mathcal{M}) = 0$ for $\mathcal{M} = \mathcal{L}(0, 0)$.

2. $\mathcal{A} = \mathcal{L}(\mathbf{e}, \mathbf{e-1}, \mathbf{1^{2e}})$ **$\mathbf{e} \geq \mathbf{1}$ and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^{2e}})$

   Then follows $\mathcal{M} = \mathcal{L}(d - \mu \cdot e, m_0 - \mu \cdot e + \mu, (6-\mu)^{2e})$ and $\mathcal{M}.\mathcal{A} = 0$ gives $-e \cdot m_0 + e \cdot d - 12e + m_0 + \mu = 0 \Longrightarrow m_0 > d - 12$. $v(\mathcal{M}) \geq 0$ gives $d \geq m_0 + \mu - 2$.

   If $\underline{\mu = 2, 3, 4, 5}$, one needs to go through all the cases for m_0 as above.

   For $\underline{\mu = 6}$ we have that $d - 4 \geq m_0 > d - 12$. Let $m_0 = d - x$. From $\mathcal{M}.\mathcal{A} = 0 \Rightarrow d = (12-x)e + (x-6)$. We notice that $\mathcal{M} = \mathcal{L}((6-x)e + (x-6), (6-x)e, 0)$, which is regular. Taking into account that $v(\mathcal{M}) \leq -1$ for all $x \leq 5$ and $m_0 \leq -1$ for all $x \geq 7$ we get the only case:

   $m_0 = d - 6$ and $\mathcal{M}.\mathcal{A} = 0 \Rightarrow d = 6e$ and $\mathcal{M} = \mathcal{L}(0, 0)$ is regular with $v(\mathcal{M}) = 0$.

3. $\mathcal{A} = \mathcal{L}(\mathbf{e}, \mathbf{e}, \mathbf{1^{e}})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^{e}})$

   This leads to $\mathcal{M} = \mathcal{L}(d - \mu e, m_0 - \mu e, (6-\mu)^e)$. $\mathcal{M}.\mathcal{A} = 0$ gives $m_0 = d + \mu - 6$.

   If $\underline{\mu = 2}$ then we get $m_0 = d - 4$, $\mathcal{L} = \mathcal{L}(d, d-4, 6^e)$ and $\mathcal{M} = \mathcal{L}(d - 2e, d - 4 - 2e, 4^e)$. From $v(\mathcal{M}) = -20e + 5d - 6 \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq \frac{6+20e}{5}$. Further $\mathcal{M}$ is irregular by [9] and of higher dimension if

   (a) $e = 2f$ and $d = 8f$
   (b) $e = 2f$ and $d = 8f + 1$
   (c) $e = 2f$ and $d = 8f + 2$.

   For $\underline{\mu = 3}$, $\underline{\mu = 4}$, $\underline{\mu = 5}$ and $\underline{\mu = 6}$ we make similar examinations.

   The following two cases are easier to compute because we have no further parameters in the (-1)-curves.

4. $\mathcal{A} = \mathcal{L}(\mathbf{6}, \mathbf{3}, \mathbf{2^{7}})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^{7}})$, $\underline{\mu = 2, 3}$
5. $\mathcal{A} = \mathcal{L}(\mathbf{3}, \mathbf{0}, \mathbf{2^{3}})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^{3}})$, $\underline{\mu = 2, 3}$
6. $\mathcal{A} = \mathcal{L}(\mathbf{12}, \mathbf{8}, \mathbf{3^{9}})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^{9}})$, $\underline{\mu = 2}$

- $\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{2} \cdot \mathcal{A}_\mathbf{2}, \mathbf{v}(\mathcal{M}) \geq \mathbf{0}, \mathcal{M} \text{ non-special and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

1. $\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^{n}})$ **and** $\mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}$

   This leads to $\mathcal{A}_1 = \mathcal{L}(e, e-1, 1^{2e})$ and $\mathcal{A}_2 = \mathcal{L}(2e, 2e, 1^{2e})$. Further we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d - 6e, m_0 - 6e + 2, 2^{2e})$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d - 4$ and $d = 8e + 2$. Therefore we have $\mathcal{M} = \mathcal{L}(2e + 2, 2e, 2^{2e})$, which is regular by [2] and $v(\mathcal{M}) = 5$.

2. With similar considerations we treat the following case:

   $\mathcal{A}_\mathbf{1} = \mathcal{L}(\delta_\mathbf{1}, \mu_{\mathbf{0_1}}, \mathbf{1^{n}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\delta_\mathbf{2}, \mu_{\mathbf{0_2}}, \mathbf{2^{n}})$

For the rest we only mention the missing cases, which are all treated analogously.

- $\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{3} \cdot \mathcal{A}_\mathbf{2}, \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

  $\boxed{\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^n}) \text{ and } \mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}}$

  1. $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{e}, \mathbf{e-1}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$
  2. $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{e}, \mathbf{e-1}, \mathbf{1^{2e}})$

- $\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{4} \cdot \mathcal{A}_\mathbf{2}, \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

  $\boxed{\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^n}) \text{ and } \mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}}$

  1. $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{e}, \mathbf{e-1}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$
  2. $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{e}, \mathbf{e-1}, \mathbf{1^{2e}})$

- $\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{2} \cdot \mathcal{A}_\mathbf{2} + \mathbf{2} \cdot \mathcal{A}_\mathbf{3}, \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

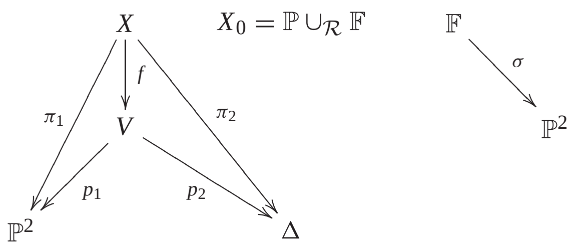
  $\mathcal{L}(e, e-1, 1^{2e})$ and $\mathcal{L}(e, e, 1^e)$ are the only quasi-homogeneous (-1)-configurations with $m_1 = \ldots = m_n = 1$ which have intersection multiplicity $= 0$. Therefore we are immediately in the previous case.

□

5. The degeneration method

In this section we give a rough overview of the degeneration of the plane as introduced by Ciliberto and Miranda in [2]. We refer to this paper for further details. As in every degeneration method the aim is to specialize the base points of a system $\mathcal{L}(d, m_0, m^n)$ in such a way that on the one hand the dimension is easier to compute but on the other hand it does not change.

At first we consider the geometric situation. Let Δ be a complex disc around the origin. We define $V = \mathbb{P}^2 \times \Delta$. Let $p_1 : V \longrightarrow \mathbb{P}^2$ and $p_2 : V \longrightarrow \Delta$ be the projections. Now we blow up a line L in $V_0 = p_2^{-1}(0)$ $(f : X \longrightarrow V)$ and obtain the following situation with $\pi_i = f \circ p_i$:



Now $X_t = \pi_2^{-1}(t) \cong \mathbb{P}^2$ for all $t \neq 0$. $X_0 = \pi_2^{-1}(0)$ is a union of two surfaces, the strict transform of $V_0 \cong \mathbb{P}^2$ (called $\mathbb{P}$) and the exceptional divisor $\mathbb{F} = f^{-1}(L)$. $\mathbb{F}$

is isomorphic to the blow-up of $\mathbb{P}^2$ in one point p (here via σ). The surfaces are glued together along the line $\mathcal{R}$, which can be identified with L in $\mathbb{P}$ and with the exceptional divisor $E = \sigma^{-1}(p)$ in $\mathbb{F}$.

As in [2] we define $\mathcal{O}_X(d) = \pi_1^*\mathcal{O}_{\mathbb{P}^2}(d)$ and $\mathcal{O}_X(d,k) = \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k\mathbb{P})$. We set $\chi(d,k) = \mathcal{O}_X(d,k)|_{X_0}$. Let H be the pull-back of a general line in $\mathbb{P}^2$ via σ. Then we have $\mathcal{O}_X(d,k)|_{X_t} \cong \mathcal{O}_{\mathbb{P}^2}(d)$ for $t \neq 0$. Furthermore $\chi(d,k)|_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}^2}(d-k)$ and $\chi(d,k)|_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(dH - (d-k)E)$.

We fix $n-b+1$ general points $p_0, p_1, \ldots, p_{n-b}$ on $\mathbb{P}$ and b general points $p_{n-b+1}, \ldots, p_n$ on $\mathbb{F}$. We define $\mathcal{L}_0$ to be the linear sub-system of $\chi(d,k)$ defined by all divisors of $\chi(d,k)$ having multiplicity at least m_0 at p_0 and at least m at the points $p_1, \ldots, p_n$ (write $\mathcal{L}_0 = \mathcal{L}(d, m_0, m^{n-b}, m^b)$). We say that $\mathcal{L}_0$ is obtained from $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ by an *(k,b)-degeneration*. $\mathcal{L}_0$ can be considered as a flat limit on X_0 of $\mathcal{L}$. By semi-continuity we obtain

$$\ell_0 = \ell(\mathcal{L}_0) \geq \ell(\mathcal{L}).$$

In particular, if $\ell_0 = e(\mathcal{L})$ then $\mathcal{L}$ is non-special.

Now $\mathcal{L}_0$ restricts on $\mathbb{P}$ to a system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-k, m_0, m^{n-b})$. Furthermore we restrict $\mathcal{L}_0$ on $\mathbb{F}$ to $\mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d-k, m^b)$ (the identification we obtain by blowing down $\mathcal{L}_{\mathbb{F}}$ to $\mathbb{P}^2$ via σ). Now we define as in [2] $\mathcal{R}_{\mathbb{P}}$ to be the linear system on $\mathcal{R}$ obtained by restricting $\mathcal{L}_{\mathbb{P}}$ to $\mathcal{R}$. We have the following exact sequence

$$0 \longrightarrow \hat{\mathcal{L}}_{\mathbb{P}} \xrightarrow{+L} \mathcal{L}_{\mathbb{P}} \xrightarrow{|_L} \mathcal{R}_{\mathbb{P}} \longrightarrow 0.$$

The kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$ consists of all divisors having L as component. So we can identify $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d-k-1, m_0, m^{n-b})$.

We analogously define $\mathcal{R}_{\mathbb{F}}$ and obtain $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d-k+1, m^b)$ (parametrising the divisors in $\mathcal{L}_{\mathbb{F}}$ which have E as a component).

Let us recall some further abbreviations from [2]:

DEFINITION 3.

$v_{\mathbb{P}} = v(\mathcal{L}_{\mathbb{P}})$, $v_{\mathbb{F}} = v(\mathcal{L}_{\mathbb{F}})$,

$\hat{v}_{\mathbb{P}} = v(\hat{\mathcal{L}}_{\mathbb{P}})$, $\hat{v}_{\mathbb{F}} = v(\hat{\mathcal{L}}_{\mathbb{F}})$,

$\ell_{\mathbb{P}} = \ell(\mathcal{L}_{\mathbb{P}})$, $\ell_{\mathbb{F}} = \ell(\mathcal{L}_{\mathbb{F}})$,

$\hat{\ell}_{\mathbb{P}} = \ell(\hat{\mathcal{L}}_{\mathbb{P}})$, $\hat{\ell}_{\mathbb{F}} = \ell(\hat{\mathcal{L}}_{\mathbb{F}})$,

$r_{\mathbb{P}} = \ell_{\mathbb{P}} - \hat{\ell}_{\mathbb{P}} - 1$, *the dimension of* $\mathcal{R}_{\mathbb{P}}$,

$r_{\mathbb{F}} = \ell_{\mathbb{F}} - \hat{\ell}_{\mathbb{F}} - 1$, *the dimension of* $\mathcal{R}_{\mathbb{F}}$.

In [2] it is shown that the associated vector spaces to $\mathcal{R}_{\mathbb{P}}$ and $\mathcal{R}_{\mathbb{F}}$ are transversal subspaces of $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}}(d-k))$. This leads to the following corollary:

COROLLARY 1 (KEY-LEMMA ON ℓ_0). *We have two cases:*

1. *If* $r_{\mathbb{P}} + r_{\mathbb{F}} \leq d - k - 1$, *then* $\ell_0 = \hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} + 1$.

2. *If* $r_{\mathbb{P}} + r_{\mathbb{F}} \geq d - k - 1$, *then* $\ell_0 = \ell_{\mathbb{P}} + \ell_{\mathbb{F}} - d + k$.

A proof can be found in [2]

6. Proof of the Main Theorem

Before giving the proof let us state two lemmas which are corollaries of the Key-Lemma 1. The proof of these is given for an analogous case in [8].

LEMMA 3 (CASE $v(\mathcal{L}) \leq -1$). *Let* $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ *with* $v(\mathcal{L}) \leq -1$. *If there are integers* k $(k < d)$ *and* b $(b < n)$ *such that a* (k, b)*-degeneration can be found with the following properties of the restrictions of* $\mathcal{L}_0$

- $\mathcal{L}_{\mathbb{F}}$ *and* $\mathcal{L}_{\mathbb{P}}$ *are both non-special, and*
- *the kernel systems* $\hat{\mathcal{L}}_{\mathbb{F}}$ *and* $\hat{\mathcal{L}}_{\mathbb{P}}$ *are empty with* $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$,

then $\mathcal{L}$ *is empty.*

LEMMA 4 (CASE $v(\mathcal{L}) \geq -1$). *Let* $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ *with* $v(\mathcal{L}) \geq -1$. *If there are integers* k $(k < d)$ *and* b $(b < n)$ *such that a* (k, b)*-degeneration can be found with*

- $\mathcal{L}_{\mathbb{F}}$ *and* $\mathcal{L}_{\mathbb{P}}$ *are both non-special,* $v_{\mathbb{P}} \geq -1$, $v_{\mathbb{F}} \geq -1$, *and*
- *the kernel systems* $\hat{\mathcal{L}}_{\mathbb{F}}$ *and* $\hat{\mathcal{L}}_{\mathbb{P}}$ *have the property* $v(\mathcal{L}) - 1 \geq \hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}}$,

then $\mathcal{L}$ *is non-special.*

The following three lemmas state parts of the result of the Main Theorem 1. We prove them independently later on.

LEMMA 5 (THREE BASE POINTS). *A linear system* $\mathcal{L}(d, m_0, m^n)$ *with at most three base points (*$n \leq 2$*) is special if and only if it is* (-1)*-special.*

LEMMA 6 (LARGE MULTIPLICITIES m_0 IN p_0). *Let* $d \geq 25$. *If* $m_0 \geq d - 9$ *then* $\mathcal{L}(d, m_0, 6^n)$ *is special if and only if it is* (-1)*-special.*

LEMMA 7 (LOW DEGREES). *If* $d \leq 140$ *then* $\mathcal{L}(d, m_0, 6^n)$ *is special if and only if it is* (-1)*-special.*

Proof of the Main Theorem 1. Let $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$. By the lemma for large multiplicities (6) we can assume that $d \geq m_0 + 10 \geq 10$.

Furthermore by the lemma for low degrees (7) the statement is true for $d \leq 140$. We can assume $d \geq 141$. We continue by induction on d where 7 can be considered as the base of the induction.

As all such $\mathcal{L}$ are not (-1)-special we have to show that $\mathcal{L}$ is non-special. The method is to get the system $\mathcal{L}_0$ on the special fiber by a degeneration of $\mathcal{L}$. With Lemmas 3 and 4 we can prove the regularity of $\mathcal{L}$ if the restrictions of $\mathcal{L}_0$ to $\mathbb{P}$ and to $\mathbb{F}$ have certain properties. These properties can be achieved as the main conjecture holds for the systems on $\mathbb{P}$ by induction and for the ones on $\mathbb{F}$ by 6.

We perform now a $(5, b)$-degeneration on $\mathcal{L}$ and get the following systems on the special fiber:

$$\mathbb{P}: \quad \mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-5, m_0, 6^{n-b}) \qquad \mathbb{F}: \quad \mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d-5, 6^b)$$

$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d-6, m_0, 6^{n-b}) \qquad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d-4, 6^b)$$

Step 1 (case $v(\mathcal{L}) \leq -1$):

We want to apply Lemma 3 for the case $v(\mathcal{L}) \leq -1$.

First of all we need to have $\hat{\mathcal{L}}_{\mathbb{F}}$ empty. By the lemma for large multiplicities in m_0 (6) we have that $\hat{\mathcal{L}}_{\mathbb{F}}$ is non-special if it is non-(-1)-special. Therefore by our classification theorem 2 it is sufficient to choose $d < 4b$, i.e., $b > \frac{d}{4}$. Also we get $\hat{v}_{\mathbb{F}} \leq -1$, which means this system is empty.

Next let us find a sufficient condition to get $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$. A computation gives $\hat{v}_{\mathbb{P}} - v(\mathcal{L}) = -6d + 21b + 9$, hence it is sufficient to have $-6d + 21b + 9 \leq 0$, that is $b \leq \frac{6d-9}{21}$.

Now we want to find sufficient conditions to have $\mathcal{L}_{\mathbb{F}}$ non-special. By 6 this is already the case if we find conditions for $\mathcal{L}_{\mathbb{F}}$ not to be (-1)-special. By Theorem 2 it is sufficient to force $d > \frac{7b}{2} + 3$, that is $b < \frac{2}{7}(d-3)$. As $\frac{2}{7}(d-3) \leq \frac{6d-9}{21}$, this new condition on b includes also $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$.

In the next step we are searching for a sufficient condition to get $\mathcal{L}_{\mathbb{P}}$ non-special. By induction on d $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-5, m_0, 6^{n-b})$ is special if and only if it is (-1)-special. By our list in Theorem 2 we notice that $\mathcal{L}_{\mathbb{P}}$ is non-(-1)-special if we choose $n-b$ odd as we have assumed that $d - m_0 \geq 10$ and $d \geq 141$.

In the last step we look for a sufficient condition on b to get $\hat{\mathcal{L}}_{\mathbb{P}}$ empty. Here we have to be more careful. When $d - m_0 \geq 11$ we get for the same reasons as in the case of $\mathcal{L}_{\mathbb{P}}$ that $\hat{\mathcal{L}}_{\mathbb{P}}$ is non-special if $n-b$ is odd. When $d - m_0 = 10$ then from Theorem 2 we know that $\hat{\ell}_{\mathbb{P}} = -20(n-b) + 5(d-6) - 6$ if $n-b$ is odd. That means we want this expression to be negative. From $\hat{\ell}_{\mathbb{P}} \leq -1 \Longleftrightarrow b \leq \frac{1}{4}(7-d) + n$ we get a sufficient condition on b. As by assumption $v(\mathcal{L}) \leq -1$, we can conclude that $v(\mathcal{L}) = 11d - 21n - 45 \leq -1$. Therefore $n \geq \frac{11d-44}{21}$. That means we can formulate the above condition on b without n (using a lower bound on n) and get $b \leq \frac{1}{4}(7-d) + \frac{11d-44}{21} = -\frac{29}{84} + \frac{23d}{84}$.

Let us now reformulate all sufficient conditions (separated for the cases $d - m_0 = 10$ and $d - m_0 > 10$) in a compact form: If $d - m_0 > 10$ we find a b such that we can apply Lemma 3 if

$$\frac{2}{7}d - \frac{6}{7} - \frac{1}{4}d > 2 \Longleftrightarrow d \geq 81.$$

If $d - m_0 = 10$ we find also a b to apply 3 if

$$-\frac{29}{84} + \frac{23}{84}d - \frac{1}{4}d > 2 \Longleftrightarrow d \geq 99.$$

Step 2 (case $v(\mathcal{L}) \geq -1$):

We want to use Lemma 4 for the case $v(\mathcal{L}) \geq -1$. Still all notations are with respect to the above $(5, b)$-degeneration.

In a first step we want to find a sufficient condition on b to get $\mathcal{L}_{\mathbb{P}}$ non-special. Exactly as in step 1 we get by induction that $\mathcal{L}_{\mathbb{P}}$ is non-special if we choose b such that $n - b$ is odd, because we assume $d - m_0 \geq 10$ and $d \geq 141$.

Next we want to find sufficient conditions on b to get the system $\mathcal{L}_{\mathbb{F}}$ non-special and $v_{\mathbb{F}} \geq -1$. By the lemma for large multiplicities (6) in m_0 we have again as above that $\mathcal{L}_{\mathbb{F}}$ is non-special if and only if it is non-(-1)-special. We conclude that we get $\mathcal{L}_{\mathbb{F}}$ non-special if we have $d > \frac{7}{2}b + 3$, that is if $b < \frac{2}{7}d - \frac{6}{7}$, by Theorem 2. As $v_{\mathbb{F}} = 6d - 21b - 10$ we see that $v_{\mathbb{F}} \geq -1$ which is equivalent to $b \leq \frac{2}{7}d - \frac{3}{7}$. Therefore the condition for getting $\mathcal{L}_{\mathbb{F}}$ non-special gives already that $v_{\mathbb{F}} \geq -1$.

From Theorem 2 we note again that $b > \frac{d}{4}$ confirms that $\hat{\mathcal{L}}_{\mathbb{F}}$ is non-special and $\hat{v}_{\mathbb{F}} \leq -1$.

Let us now consider $\hat{\mathcal{L}}_{\mathbb{P}}$: As above $\hat{\mathcal{L}}_{\mathbb{P}}$ is by induction non-special if $n-b$ is odd and $d-m_0 \geq 11$. In the case $d-m_0 \geq 11$ we force also $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$, that is $b \leq \frac{6d-9}{21}$. In the case $d-m_0 = 10$ we conclude - exactly as above - that if $n-b$ is odd $\hat{\mathcal{L}}_{\mathbb{P}}$ is non-special or $\hat{\ell}_{\mathbb{P}} = -20(n-b)+5(d-6)-6$. Therefore we force $-20(n-b)+5(d-6)-6 \leq -1$, that means $b \leq \frac{1}{4}(7-d)+n$. As we are in the case $v(\mathcal{L}) \geq -1$ we have the equation $11d-21n-45 \geq -1$ which means $n \leq \frac{11d-44}{21}$. It is enough to check the independence of all conditions on the base points in $\mathcal{L}$ for the highest possible number n of points. We fix this n and use a lower bound $\frac{11d-44}{21} - 1$ of it. That means a sufficient condition for $\hat{\mathcal{L}}_{\mathbb{P}}$ to be non-special is $b \leq \frac{1}{4}(7-d) + \frac{11d-44}{21} - 1 = \frac{-113+23d}{84}$.

To fulfill all these conditions we need to have d large enough. All together this gives so far:

If $d - m_0 \geq 11$ we are able to find a sufficient b if

$$\frac{2}{7}d - \frac{6}{7} - \frac{1}{4}d > 2 \Longleftrightarrow d \geq 81.$$

If $d - m_0 = 10$ we are able to find a sufficient b if

$$\frac{23}{84}d - \frac{113}{84} - \frac{1}{4}d > 2 \Longleftrightarrow d \geq 141.$$

In both cases we have that $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ are non-special and $v_{\mathbb{F}} \geq -1$. From $\hat{v}_{\mathbb{F}} \leq -1$ and from $v_{\mathbb{P}} = v - 1 - \hat{v}_{\mathbb{F}}$ we get immediately $v_{\mathbb{P}} \geq -1$. We have $v \geq \hat{v}_{\mathbb{P}}$. As $\hat{\mathcal{L}}_{\mathbb{F}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ are non-special we are able to conclude the following two cases:

If $\hat{v}_{\mathbb{P}} \leq -1$ then

$$\hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} = -2 \leq v - 1,$$

and if $\hat{v}_{\mathbb{P}} \geq -1$ then

$$\hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} = \hat{v}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} \leq v - 1.$$

In both cases we are able to apply Lemma 4 and conclude that $\mathcal{L}$ is non-special.

□

7. Proof of the Lemmas

Before starting the proofs we should take some time to explain the use of Quadratic Cremona Transformations for our purpose. We identify such a transformation with blowing up three general points and blowing down their connecting lines. Such a transformation is called to be *based on the three points*. Furthermore one can see by the blow-up and -down interpretation that a linear system $\mathcal{L}(d, m_0, m_1, m_2, m_3, \ldots, m_n)$ is transformed by a Cremona transformation based on the points p_0, p_1, p_2 to a system $\mathcal{L}(2d - m_0 - m_1 - m_2, d - m_1 - m_2, d - m_0 - m_2, d - m_0 - m_1, m_3, \ldots, m_n)$. If all involved numbers are non-negative (see [2]), the dimension and the virtual dimension of a system $\mathcal{L}$ do not change under Cremona transformations. In fact a (-1)-curve splitting off a system $\mathcal{L}$ is transformed again into a (-1)-curve, which splits off the transformed system. Therefore it is equivalent to examine a system $\mathcal{L}$ or its Cremona transformed for our purpose. We use suitable sequences of Cremona transformations in the following proofs to obtain systems which are already examined in previous papers.

Proof of the lemma of three base points 5. This can be seen by direct computations with base points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Of course, the statement is also included in the result in [5]. □

Proof of the lemma of large multiplicities m_0 in p_0 6. We consider the system $\mathcal{L}(d, m_0, 6^n)$. For the case of $m_0 \geq d - 7$ [2, Proposition 6.2., Corollary 6.3., Proposition 6.4.] give a classification of the special systems of this type. Comparing it with our list in Theorem 2 gives the statement. Now let $d \geq 25$. The strategy for the proof is to perform a sequence of Cremona transformations in order to get systems, which can be examined easier. Furthermore we apply the degeneration method again and use again Cremona transformations to prove regularity of some of the obtained systems.

case: $d - m_0 = 8$

Let $\mathcal{L} = \mathcal{L}(d, d - 8, 6^n)$. We note that if we perform k Cremona transformations, based on p_0 and successively on two other base points of multiplicity 6, we obtain that it is now equivalent to consider the Cremona transformed system (for the strategy see [8]):

$$\mathcal{L} \sim \mathcal{L}(d - 4k, d - 8 - 4k, 6^{n-2k}, 2^{2k})$$

We set $d - 8 = 4t + \epsilon$ with $\epsilon \in \{0, 1, 2, 3\}$. And $n = 2q + \eta$ with $\eta \in \{0, 1\}$.

If $t \leq q$ we perform $k = t$ transformations on $\mathcal{L}(d, d-8, 6^n)$ based on p_0 and successively two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(8+\epsilon, \epsilon, 6^{n-2t}, 2^{2t}).$$

The system on the right hand side is of bounded multiplicity, that means all multiplicities are ≤ 6. Such systems are special if and only if they are (-1)-special by [10].

If $t > q$ we perform $k = q$ transformations on $\mathcal{L}(d, d-8, 6^n)$ again based on p_0 and successively two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(d-4q, d-8-4q, 6^{\eta}, 2^{2q}).$$

If $\eta = 0$ we are in the case of quasi-homogeneous linear systems of multiplicity 2, here the main conjecture is true by [2].

If $\eta = 1$ we have to examine systems of the type $\mathcal{L} = \mathcal{L}(\delta, \delta-8, 6, 2^{2q})$ with $\delta = d - 4q$. Now let us perform a $(2, b)$-degeneration and get the following systems:

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta-2, \delta-8, 6, 2^{2q-b}) \;\; \mathcal{L}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-2, 2^b)$$
$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta-3, \delta-8, 6, 2^{2q-b}) \;\; \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-1, 2^b)$$

If $v(\mathcal{L}) \leq -1$ we want to apply lemma 3.

By our classification Theorem 2 there is no (-1)-special system of the type $\mathcal{L}(d, d-8, 6^n)$ if $d \geq 25$. That means we have to show that the system $\mathcal{L}$ is empty. To use 3 we have again to consider all the systems obtained by the degeneration as in the proof of the main theorem.

In a first step let us consider $\hat{\mathcal{L}}_{\mathbb{F}}$. As $\hat{\mathcal{L}}_{\mathbb{F}}$ is a quasi-homogeneous system of multiplicity $m = 2$ we see in [2], that this system is never special. Then $\hat{v}_{\mathbb{F}} = 2\delta - 3b$ leads to a sufficient condition to get $\hat{\mathcal{L}}_{\mathbb{F}}$ empty. This condition is $b \geq \frac{2\delta+1}{3}$.

In a next step we want to find a sufficient condition to get $\mathcal{L}_{\mathbb{F}}$ non-special. This is true by [2] if b is odd. So let us force b to be odd as a sufficient condition for this case.

Now we consider $\mathcal{L}_{\mathbb{P}}$. We claim: $\mathcal{L}_{\mathbb{P}}$ is non-special.

To show the claim we apply at first a Cremona transformation based on the points of multiplicity $\delta - 8$, 6 and on one point of multiplicity 2. This leads to the following system:

$$\mathcal{L}_{\mathbb{P}} \sim \mathcal{L}(\delta-4, \delta-10, 4, 2^{2q-b-1}).$$

Above we forced b to be odd, therefore we assume $2q - b - 1 \geq 2$ (otherwise skip this step) is even. Now we apply successively $\frac{2q-b-1}{2}$ Cremona transformations, based in p_0 and two points of multiplicity 2. Therefore we see that we have the following equivalence:

$$\mathcal{L}_{\mathbb{P}} \sim \mathcal{L}(\delta-4+2q-b-1, \delta-10+2q-b-1, 4^{2q-b}).$$

From $\delta = d - 4q \geq 12 + \epsilon$ we get by [9, Theorem 2.1, Theorem 5.2] that this system is never special.

Finally we have to consider $\hat{\mathcal{L}}_{\mathbb{P}}$. Again we claim that $\hat{\mathcal{L}}_{\mathbb{P}}$ is never special.

We have by the above assumption that $2q - b$ is odd. At first we split off the line through the points of multiplicity $\delta - 8$ and 6. As the virtual dimension doesn't change we get

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta - 4, \delta - 9, 5, 2^{2q-b}).$$

Another Cremona transformation based in p_0, p_1 and one point of multiplicity 2 leads to the equivalence

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta - 6, \delta - 11, 3, 2^{2q-b-1}).$$

Now as in the case of $\mathcal{L}_{\mathbb{P}}$ we apply another $\frac{2q-b-1}{2}$ Cremona transformations based in p_0 and successively in two points of multiplicity 2. We end up with the equivalence:

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta - 6 + \frac{2q-b-1}{2}, \delta - 11 + \frac{2q-b-1}{2}, 3^{2q-b}).$$

Now we are able to conclude with [2] - as we are in the case of a quasi-homogeneous system of multiplicity 3 - that this system is never special.

To apply 3 we have to find a sufficient condition for b to get $\hat{v}_{\mathbb{P}} \leq -1$, therefore it is sufficient to have $\hat{v}_{\mathbb{P}} - v(\mathcal{L}) \leq 0$, which is equivalent to $b \leq \delta$.

All together we find a sufficient b if $\delta - \frac{2\delta+1}{3} \geq 2 \iff \delta \geq 8$. As we have seen above we have already $\delta \geq 12 + \epsilon$. This means we can apply Lemma 3 and conclude that $\mathcal{L}(d, d-8, 6^n)$ is empty in the case $v(\mathcal{L}) \leq -1$.

Now we have to consider the case $v(\mathcal{L}) \geq -1$. Here we want to apply the Lemma 4.

As in the case $v(\mathcal{L}) \leq -1$ we can always find a b such that all the systems obtained by the above $(2, b)$-degeneration are non-special. Let us choose such a b like above and then consider the systems $\mathcal{L}_{\mathbb{P}}$, $\hat{\mathcal{L}}_{\mathbb{P}}$, $\mathcal{L}_{\mathbb{F}}$ and $\hat{\mathcal{L}}_{\mathbb{F}}$. From $v_{\mathbb{P}} = v(\mathcal{L}) - \hat{v}_{\mathbb{F}} - 1$, $\hat{v}_{\mathbb{F}} \leq -1$ and $v(\mathcal{L}) \geq -1$ we conclude $v_{\mathbb{P}} \geq v(\mathcal{L}) \geq -1$. A direct computation gives $v_{\mathbb{F}} \geq -1$.

As the inequality $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$ is also fulfilled we get $\hat{\ell}_{\mathbb{P}} \leq v(\mathcal{L})$. Therefore we can apply Lemma 4 and conclude that $\mathcal{L}(d, d-8, 6^n)$ is non-special.

<u>case: $d - m_0 = 9$</u>

Let $\mathcal{L} = \mathcal{L}(d, d-9, 6^n)$. We note as above that if we perform k Cremona transformations, based on p_0 and successively on two other base points of multiplicity 6, we obtain that:

$$\mathcal{L} \sim \mathcal{L}(d - 3k, d - 9 - 3k, 6^{n-2k}, 3^{2k})$$

We set $d - 9 = 3t + \epsilon$ with $\epsilon \in \{0, 1, 2\}$. And $n = 2q + \eta$ with $\eta \in \{0, 1\}$.

If $t \leq q$ we perform $k = t$ transformations on $\mathcal{L}(d, d-9, 6^n)$ based on m_0 and successively on two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(9 + \epsilon, \epsilon, 6^{n-2t}, 3^{2t}).$$

Then the system on the right hand side is of bounded multiplicity, that means all multiplicities are ≤ 6. As mentioned above such systems are *special* if and only if they are *(-1)-special* by [10].

If $t > q$ we perform $k = q$ transformations on $\mathcal{L}(d, d-9, 6^n)$ and obtain

$$\mathcal{L} \sim \mathcal{L}(d-3q, d-9-3q, 6^\eta, 3^{2q}).$$

If $\eta = 0$ we are in the case of quasi-homogeneous linear systems of multiplicity 3, here the main conjecture is true by [2].

If $\eta = 1$ we have to examine systems of the type $\mathcal{L}(\delta, \delta-9, 6, 3^{2q})$ with $\delta = d-3q$. If $\delta < 15$ we are in the case of systems of bounded multiplicity where the main conjecture holds by [10]. So we can assume $\delta \geq 15$. Also we can assume $q \geq 1$ (otherwise the statement is clear). Now let us perform a $(3, b)$-degeneration and get the following systems:

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta-3, \delta-9, 6, 3^{2q-b}) \quad \mathcal{L}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-3, 3^b)$$

$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta-4, \delta-9, 6, 3^{2q-b}) \quad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-2, 3^b)$$

It turns out that we can apply lemmas 3 and 4 as above if $\delta \geq 15$. Especially we see again by applying Cremona transformations that $\mathcal{L}_{\mathbb{P}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ are both non-special (for details see [6]). On the other hand as $\delta \geq 15$ we find a b such that $\hat{\mathcal{L}}_{\mathbb{F}}$ is empty and $\mathcal{L}_{\mathbb{F}}$ is non-special.

□

Proof of the lemma of low degrees 7. The main tool for this proof is a computer program which uses $(5, b)$- and $(6, b)$-degenerations of the plane in order to prove that certain non-(-1)-special systems are non-special. This algorithm is given by Laface and Ugaglia in [8]. We implemented this algorithm in *Singular* (see [4]). Furthermore to treat the cases where the degeneration-method fails we implemented a method used by Yang in [10]. This method specializes the base points on a line and moves them to infinity. Then it is easier to check if the given conditions on the base points are independent. If this is still the case it proves regularity of a given system.
Below we list only the cases in which the program fails. All these but 10 cases are solved by ad-hoc methods (mainly Cremona transformations). The remaining 10 cases we computed directly with *Singular* in characteristic 32003. One can see that this implies then regularity in characteristic 0, too.

$d-m_0$	system	dim.	method
8	$\mathcal{L}=\mathcal{L}(8,0,6^3)$	-1	3-point lemma
8	$\mathcal{L}=\mathcal{L}(9,1,6^3)$	-1	splitting off lines
14	$\mathcal{L}=\mathcal{L}(14,0,6^6)$	-1	Cremona, splitting off lines
13	$\mathcal{L}=\mathcal{L}(14,1,6^6)$	-1	as $\mathcal{L}(14,0,6^6)$ is empty
12	$\mathcal{L}=\mathcal{L}(14,2,6^6)$	-1	as $\mathcal{L}(14,0,6^6)$ is empty
11	$\mathcal{L}=\mathcal{L}(14,3,6^6)$	-1	as $\mathcal{L}(14,0,6^6)$ is empty
10	$\mathcal{L}=\mathcal{L}(14,4,6^6)$	-1	as $\mathcal{L}(14,0,6^6)$ is empty
8	$\mathcal{L}=\mathcal{L}(14,6,6^5)$	-1	as $\mathcal{L}(14,0,6^6)$ is empty
15	$\mathcal{L}=\mathcal{L}(15,0,6^7)$	-1	Cremona
15	$\mathcal{L}=\mathcal{L}(15,0,6^6)$	>-1	as $\mathcal{L}(15,3,6^6)$ is regular
14	$\mathcal{L}=\mathcal{L}(15,1,6^6)$	>-1	as $\mathcal{L}(15,3,6^6)$ is regular
13	$\mathcal{L}=\mathcal{L}(15,2,6^6)$	>-1	as $\mathcal{L}(15,3,6^6)$ is regular
12	$\mathcal{L}=\mathcal{L}(15,3,6^6)$	>-1	Cremona and [2]
11	$\mathcal{L}=\mathcal{L}(15,4,6^6)$	-1	Cremona, splitting off lines
10	$\mathcal{L}=\mathcal{L}(15,5,6^6)$	-1	as $\mathcal{L}(15,4,6^6)$ is empty
9	$\mathcal{L}=\mathcal{L}(15,6,6^6)$	-1	as $\mathcal{L}(15,4,6^6)$ is empty
9	$\mathcal{L}=\mathcal{L}(15,6,6^5)$	>-1	as $\mathcal{L}(15,0,6^6)$ is regular
8	$\mathcal{L}=\mathcal{L}(15,7,6^5)$	>-1	Cremona and [2]
16	$\mathcal{L}=\mathcal{L}(16,0,6^8)$	-1	as $\mathcal{L}(16,3,6^7)$ is empty
16	$\mathcal{L}=\mathcal{L}(16,0,6^7)$	>-1	as $\mathcal{L}(16,2,6^7)$ is regular
15	$\mathcal{L}=\mathcal{L}(16,1,6^7)$	>-1	as $\mathcal{L}(16,2,6^7)$ is regular
14	$\mathcal{L}=\mathcal{L}(16,2,6^7)$	>-1	Cremona and [2]
13	$\mathcal{L}=\mathcal{L}(16,3,6^7)$	-1	Cremona, splitting off lines
12	$\mathcal{L}=\mathcal{L}(16,4,6^7)$	-1	as $\mathcal{L}(16,3,6^7)$ is empty
11	$\mathcal{L}=\mathcal{L}(16,5,6^7)$	-1	as $\mathcal{L}(16,3,6^7)$ is empty
10	$\mathcal{L}=\mathcal{L}(16,6,6^7)$	-1	as $\mathcal{L}(16,3,6^7)$ is empty
10	$\mathcal{L}=\mathcal{L}(16,6,6^6)$	>-1	as $\mathcal{L}(16,2,6^7)$ is regular
9	$\mathcal{L}=\mathcal{L}(16,7,6^6)$	-1	Cremona, splitting off lines
8	$\mathcal{L}=\mathcal{L}(16,8,6^6)$	-1	as $\mathcal{L}(16,7,6^6)$ is empty
17	$\mathcal{L}=\mathcal{L}(17,0,6^8)$	>-1	as $\mathcal{L}(17,1,6^8)$ is regular
16	$\mathcal{L}=\mathcal{L}(17,1,6^8)$	>-1	Cremona
15	$\mathcal{L}=\mathcal{L}(17,2,6^8)$	-1	Cremona, splitting off lines
11	$\mathcal{L}=\mathcal{L}(17,6,6^7)$	>-1	as $\mathcal{L}(17,1,6^8)$ is regular
10	$\mathcal{L}=\mathcal{L}(17,7,6^7)$	-1	Cremona, splitting off lines
9	$\mathcal{L}=\mathcal{L}(17,8,6^7)$	-1	as $\mathcal{L}(17,7,6^7)$ is empty
8	$\mathcal{L}=\mathcal{L}(18,10,6^7)$	-1	Cremona, splitting off lines
19	$\mathcal{L}=\mathcal{L}(19,0,6^{10})$	-1	[3]
18	$\mathcal{L}=\mathcal{L}(19,1,6^{10})$	-1	as $\mathcal{L}(19,0,6^{10})$ is empty
17	$\mathcal{L}=\mathcal{L}(19,2,6^{10})$	-1	as $\mathcal{L}(19,0,6^{10})$ is empty
15	$\mathcal{L}=\mathcal{L}(19,4,6^9)$	>-1	as $\mathcal{L}(19,5,6^9)$ is regular
14	$\mathcal{L}=\mathcal{L}(19,5,6^9)$	>-1	regular by [10]
13	$\mathcal{L}=\mathcal{L}(19,6,6^9)$	-1	as $\mathcal{L}(19,0,6^{10})$ is empty
12	$\mathcal{L}=\mathcal{L}(19,7,6^9)$	-1	as $\mathcal{L}(19,0,6^{10})$ is empty

$d-m_0$	system	dim.	method
9	$\mathcal{L}=\mathcal{L}(19,10,6^7)$	>-1	Cremona and [3]
8	$\mathcal{L}=\mathcal{L}(19,11,6^7)$	-1	Cremona, splitting off lines
12	$\mathcal{L}=\mathcal{L}(20,8,6^9)$	>-1	direct computation*
11	$\mathcal{L}=\mathcal{L}(20,9,6^9)$	-1	Cremona and [8]
8	$\mathcal{L}=\mathcal{L}(20,12,6^7)$	>-1	Cremona and [3]
11	$\mathcal{L}=\mathcal{L}(21,10,6^9)$	>-1	Cremona and [10]
10	$\mathcal{L}=\mathcal{L}(21,11,6^9)$	-1	Cremona and [10]
9	$\mathcal{L}=\mathcal{L}(21,12,6^8)$	>-1	Cremona and [3]
8	$\mathcal{L}=\mathcal{L}(21,13,6^8)$	-1	Cremona, split. off lines, [3]
22	$\mathcal{L}=\mathcal{L}(22,0,6^{13})$	>-1	as $\mathcal{L}(22,1,6^{13})$ is regular
21	$\mathcal{L}=\mathcal{L}(22,1,6^{13})$	>-1	[10]
20	$\mathcal{L}=\mathcal{L}(22,2,6^{13})$	-1	[10]
19	$\mathcal{L}=\mathcal{L}(22,3,6^{13})$	-1	as $\mathcal{L}(22,2,6^{13})$ is empty
16	$\mathcal{L}=\mathcal{L}(22,6,6^{12})$	>-1	as $\mathcal{L}(22,1,6^{13})$ is regular
15	$\mathcal{L}=\mathcal{L}(22,7,6^{12})$	-1	direct computation *
13	$\mathcal{L}=\mathcal{L}(22,9,6^{11})$	-1	direct computation *
11	$\mathcal{L}=\mathcal{L}(22,11,6^{10})$	-1	Cremona and [10]
10	$\mathcal{L}=\mathcal{L}(22,12,6^{10})$	-1	as $\mathcal{L}(22,11,6^{10})$ is empty
10	$\mathcal{L}=\mathcal{L}(22,12,6^9)$	>-1	Cremona and [9]
9	$\mathcal{L}=\mathcal{L}(22,13,6^9)$	-1	Cremona, splitting off lines
8	$\mathcal{L}=\mathcal{L}(22,14,6^9)$	-1	as $\mathcal{L}(22,13,6^9)$ is empty
12	$\mathcal{L}=\mathcal{L}(23,11,6^{11})$	>-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(23,13,6^{10})$	-1	Cremona and [9]
9	$\mathcal{L}=\mathcal{L}(23,14,6^9)$	>-1	Cremona and [3]
8	$\mathcal{L}=\mathcal{L}(23,15,6^9)$	-1	Cremona and splitting off lines
10	$\mathcal{L}=\mathcal{L}(24,14,6^{10})$	>-1	Cremona and [3]
9	$\mathcal{L}=\mathcal{L}(24,15,6^{10})$	-1	Cremona and [10]
8	$\mathcal{L}=\mathcal{L}(24,16,6^{10})$	-1	as $\mathcal{L}(24,15,6^{10})$ is empty
13	$\mathcal{L}=\mathcal{L}(25,12,6^{13})$	-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(25,15,6^{11})$	-1	Cremona and [10]
12	$\mathcal{L}=\mathcal{L}(26,14,6^{13})$	-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(29,19,6^{13})$	>-1	direct computation *
13	$\mathcal{L}=\mathcal{L}(31,18,6^{17})$	-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(31,21,6^{14})$	>-1	Cremona and [9]
10	$\mathcal{L}=\mathcal{L}(38,28,6^{18})$	-1	Cremona and [9]
13	$\mathcal{L}=\mathcal{L}(40,27,6^{23})$	-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(40,30,6^{19})$	-1	direct computation *
10	$\mathcal{L}=\mathcal{L}(46,36,6^{22})$	-1	Cremona and [9]

□

*with [4] in char = 32003

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FREE RESOLUTIONS OF FAT POINTS ON A CONIC IN $\mathbb{P}^2$

Abstract. This paper is concerned with free resolutions of fat points in $\mathbb{P}^2$, respectively in some blown up surface of $\mathbb{P}^2$. A short overview is given about methods developed mainly by Harbourne, working with linear systems of divisors on a blown up surface. Finally, these methods are used to compute free resolutions in concrete terms for two cases: for the case that all points lie on a smooth conic and the highest multiplicity occurs four times, and for the case that all but one point lie on a line, and the one point has multiplicity one.

1. Introduction

We want to compute free resolutions of fat point ideals in $\mathbb{P}^2$. Let $p_1, \ldots, p_n$ be points in $\mathbb{P}^2$. Associate to every point p_i a multiplicity m_i. By a fat point ideal, we denote an ideal I of codimension 2 in $R := K[x, y, z]$, the coordinate ring of $\mathbb{P}^2$, where I contains all curves passing through the given points with the given multiplicities, i.e. $I = \mathfrak{m}_{p_1}^{m_1} \cap \ldots \cap \mathfrak{m}_{p_n}^{m_n}$, where $\mathfrak{m}_{p_i}$ denotes the vanishing ideal of p_i. (Later on, we will also allow infinitely near points.) Denote by h_I the Hilbert function of an ideal I. Certainly, the Hilbert function is related to the free resolution of I. We will see that the knowledge of the Hilbert function and some other properties of I will indeed be sufficient to describe the free resolution. If we blow up $\mathbb{P}^2$ at the n points, we get a rational surface X whose divisor class group is generated by $E_0, E_1, \ldots, E_n$, where E_0 denotes the pullback of a line in $\mathbb{P}^2$ and E_i denotes the exceptional divisor of the point p_i for $i = 1, \ldots, n$. These generators fulfill the relations $E_0^2 = 1$, $E_i^2 = -1$ for $i > 0$ and $E_i.E_j = 0$ for $i \neq j$. Then

$$h_I(d) = h^0(X, \mathcal{O}_X(dE_0 - m_1E_1 - \ldots - m_nE_n)$$

(see 3.2). Also, we will see that the other properties of I that are needed in order to give the free resolution can be understood in terms of linear systems of divisors on the blown up surface X. So most of the work that needs to be done is about linear systems on rational surfaces. Most people working on fat point ideals are rather interested in points in general position. However, for points in general position, the method of blowing up and working with linear systems of divisors does not in general give a solution to the problem. But if we put the points in a special position, the method allows to compute free resolutions. Harbourne showed in [5] that a free resolution can be calculated if the points lie on a conic. In this paper, we want to give a short overview about Harbourne's method. We will use the theory to calculate a free resolution explicitly for two different cases: For the case that $n - 1$ points lie on a line, and the n-th point has multiplicity 1, and for the case that all points lie on a smooth conic, and the highest multiplicity occurs four times. The results are presented in the following Theorems 1 and 2. We will give a detailed proof of Theorem 2, referring to [6] for the proof of Theorem 1, which works analogously.

THEOREM 1. *Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal. Let $(p_1, \dots, p_n)$ be a representative such that $m_1 \geq \dots \geq m_n$. Assume $m_n = 1$. Let p_1 lie on a line $L \subset \mathbb{P}^2$ and let p_i lie on the strict transform of L in $\mathrm{Bl}_{p_{i-1}} \dots \mathrm{Bl}_{p_1} \mathbb{P}^2$ for all $i = 2, \dots, n-1$. (For an explanation of this notation, see chapter 2.) Then the free resolution I is: if $m_1 > m_2$ and $\mu_{m_2} = 1$:*

$$0 \longrightarrow R[-(m_1+1)]^{m_1-m_2-1} \oplus R[-(m_1+2)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-m_1]^{m_1-m_2} \oplus R[-(m_1+1)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\ ;$$

if $m_1 > m_2$ and $\mu_{m_2} > 1$:

$$0 \longrightarrow R[-(m_1+1)]^{m_1-m_2-1} \oplus R[-(m_1+2)] \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-m_1]^{m_1-m_2} \oplus R[-(m_1+1)] \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\ ;$$

if $m_1 = m_2$ and $\mu_{m_2} = 1$:

$$0 \longrightarrow R[-(m_1+2)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-(m_1+1)]^3 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\ ;$$

if $m_1 = m_2$ and $\mu_{m_2} > 1$:

$$0 \longrightarrow R[-(m_1+2)] \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-(m_1+1)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\ ;$$

where a_i denotes the special degrees of I, $a_i = m_1 + \mu_{m_2} + \dots + \mu_i$. (For a definition of special degree and μ_i, see page 75 resp. definition 2.)

THEOREM 2. *Let $I = I(\mathcal{K}, \underline{v})$ be a fat point ideal. Asume that the highest multiplicity, m, i.e. the highest entry in the vector $\underline{v}$, occurs four times in $\underline{v}$. Choose a representative $(p, q, r, s, p_1, \dots, p_n)$ in $\mathcal{K}$ such that the highest multiplicity m is associated to p, q, r and s and such that for the other associated multiplicities, $m_1 \geq \dots \geq m_n$ holds. Assume that p, q, r, s and $p_1, \dots, p_n$ lie on an irreducible conic, respectively on the strict transform of the conic. (For more explanations, see 2 and 5.) Then the free resolution of the ideal I:*

1. *If $c := m_1 - m_2$ is even, then the free resolution is*

$$0 \to \begin{array}{c} R[-(2m+2)]^{m-m_1-1} \\ \oplus \\ \bigoplus_{a=2}^{c/2+1} R[-(2m+a)]^3 \\ \oplus \\ R[-(2m+c/2+2)] \\ \oplus \\ \bigoplus_{i=0}^{m_2-1} \left(\begin{array}{c} R[-(2m+\lceil\frac{a_{m_2-i}}{2}\rceil+1)]^{j_i} \\ \oplus \\ R[-(2m+\lceil\frac{a_{m_2-i}}{2}\rceil+2)]^{l_i} \end{array} \right) \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} R[-2m]^{m-m_1+1} \\ \oplus \\ \bigoplus_{a=1}^{c/2} R[-(2m+a)]^3 \\ \oplus \\ \bigoplus_{i=0}^{m_2-1} R[-(2m+\frac{a_{m_2-i}}{2})]^{k_i} \end{array} \longrightarrow I \to 0.$$

2. *If $c := m_1 - m_2$ is odd, then the free resolution is:*

$$0 \to \begin{array}{c} R[-(2m+2)]^{m-m_1-1} \\ \oplus \\ \bigoplus_{a=2}^{(c-1)/2+1} R[-(2m+a)]^3 \\ \oplus \\ R[-(2m+(c+1)/2+1)]^3 \\ \oplus \\ \bigoplus_{i=0}^{m_2-1} \left(\begin{array}{c} R[-(2m+\lceil\frac{a_{m_2-i}}{2}\rceil+1)]^{j_i} \\ \oplus \\ R[-(2m+\lceil\frac{a_{m_2-i}}{2}\rceil+2)]^{l_i} \end{array} \right) \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} R[-2m]^{m-m_1+1} \\ \oplus \\ \bigoplus_{a=1}^{(c-1)/2} R[-(2m+a)]^3 \\ \oplus \\ R[-(2m+(c+1)/2)]^2 \\ \oplus \\ \bigoplus_{i=0}^{m_2-1} R[-(2m+\frac{a_{m_2-i}}{2})]^{k_i} \end{array} \longrightarrow I \to 0;$$

where

$$(k_i, l_i, j_i) := \begin{cases} (1,1,0) & \text{if } a_{m_2-i} \text{ is even} \\ (2,0,2) & \text{if } a_{m_2-i} \text{ is odd} \end{cases}$$

and $a_{m_2-i} = c + i + 1 + \mu_{m_2} + \ldots + \mu_{m_2-i}$. (For a definition of a_i and μ_i, see page 75 resp. definition 2 or page 76.)

2. Infinitely near points

As already said in the introduction, we wish to allow infinitely near points, too. That is, we choose $p_1 \in \mathbb{P}^2$, $p_2 \in \mathrm{Bl}_{p_1}\mathbb{P}^2$ (where $\mathrm{Bl}_{p_1}\mathbb{P}^2$ denotes the blow up of $\mathbb{P}^2$ at the point p_1) and so on. Let $(m_1, \ldots, m_n)$ be the corresponding multiplicities. If the points were all in $\mathbb{P}^2$, we could enumerate them in such a way that $m_1 \geq \ldots \geq m_n$. As we allow infinitely near points, reordering does not make any sense, since the points live in different surfaces. But later on, we need the multiplicities to be ordered. If, for example, $p_1 \in \mathbb{P}^2$ and $p_2 \in \mathrm{Bl}_{p_1}\mathbb{P}^2 \setminus E_1$, it does not really make a difference if we consider the tuple (p_1, p_2) or (p_2, p_1). If $m_2 > m_1$, we would then choose the second possibility. So we try to group tuples of points to equivalence classes and we try to find a representative in each equivalence class where the points are ordered in such a way that $m_1 \geq \ldots \geq m_n$. Let $\pi_i : \mathrm{Bl}_{p_i} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 \to Bl_{p_{i-1}} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$ denote the map of blowing up the point p_i. Then two tuples $(p_1, \ldots, p_n)$ and $(p_1', \ldots, p_n')$ will be called equivalent if there exists an isomorphism $\Phi : \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 \to \mathrm{Bl}_{p_n'} \ldots \mathrm{Bl}_{p_1'}\mathbb{P}^2$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 & \xrightarrow{\Phi} & \mathrm{Bl}_{p_n'} \ldots \mathrm{Bl}_{p_1'}\mathbb{P}^2 \\
\downarrow \pi_n & & \downarrow \pi_n' \\
\mathrm{Bl}_{p_{n-1}} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 & & \mathrm{Bl}_{p_{n-1}'} \ldots \mathrm{Bl}_{p_1'}\mathbb{P}^2 \\
\downarrow \pi_{n-1} & & \downarrow \pi_{n-1}' \\
\ldots & & \ldots \\
\downarrow \pi_1 & & \downarrow \pi_1' \\
\mathbb{P}^2 & = & \mathbb{P}^2
\end{array}$$

An equivalence class of a tuple $(p_1, \ldots, p_n)$ is called a *constellation*. As before, let $X = \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$, then the pullback of a line E_0 together with the total transforms of the points p_i - which we denote as before with E_i - give a free generating system of $\mathrm{Pic}(X)$. Also, $E_0^2 = 1$ and $E_i^2 = -1$. Denote by $\widehat{E_i}$ the strict transform in X of the exceptional divisor in $\mathrm{Bl}_{p_i} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$ of a point p_i.

DEFINITION 1. *1. Let $p_j \neq p_i$ be points in a constellation. We say that p_j is* proximate *to p_i, denoted by $p_j \exists\exists\mathcal{K}\ p_i$, if p_j is infineteley near to p_i.*

2. A cluster *on $\mathbb{P}^2$ is a pair $(\mathcal{K}, \underline{m})$ consisting of a constellation $\mathcal{K} = [(p_1, \ldots, p_n)]$ and an integer vector $\underline{m} = (m_1, \ldots, m_n)$. We inductively introduce the* total multiplicities *$\widehat{m_i} := m_i + \sum_{p_i \exists\exists\mathcal{K} p_j} m_j$. The cluster is called nontrivial if $\mathcal{K} \neq \emptyset$ and if there is an i with $\widehat{m_i} > 0$.*

3. A cluster $(\mathcal{K}, \underline{m})$ satisfies the proximity relations at p_i if and only if $m_i \geq \sum_{p_j \exists\exists\mathcal{K} p_i} m_j$.

4. *Let $(\mathcal{K}, \underline{m})$ be a non empty cluster on $\mathbb{P}^2$ satisfying the proximity relations. Then the ideal corresponding to $(\mathcal{K}, \underline{m})$, $I(\mathcal{K}, \underline{m}) \subset K[x_0, x_1, x_2]$, is defined by*

$$I(\mathcal{K}, \underline{m}) := \{ f \in K[x_0, x_1, x_2] \mid \operatorname{mult}_{p_i} C_i \geq \widehat{m_i} \}$$

where $C = \{f = 0\}$ denotes the curve defined by f and C_i denotes the i-th total transform of C, $(\pi_{i-1} \circ \ldots \circ \pi_1)^*(C)$.

Now we can replace our notion of fat point ideals from the introduction: I is called a fat point ideal, if it is the ideal corresponding to a non empty cluster satisfying the proximity relations. If we would allow clusters which do not satisfy the proximity relations, it may happen, that we get the same corresponding ideal for two clusters even though the multiplicities are not equal. For example, if we take the constellation $\mathcal{K} = [(p_1, p_2)]$, where $p_1 \in \mathbb{P}^2$ and $p_2 \in E_1$, then $I(\mathcal{K}, (1, 0)) = I(\mathcal{K}, (1, -1))$. If we restrict to clusters satisfying the proximity relations, the multiplicities are uniquely determined by $I(\mathcal{K}, \underline{m})$. This is an easy consequence of [2], Theorem 3.3. This theorem states that every cluster satisfying the proximity relations and where all points are infinitely near to a single point in $\mathbb{P}^2$ can be realized as the cluster that appears if we resolve a singularity of a curve. That is, for such a cluster, we can find a representative $(p_1, \ldots, p_n)$ in the constellation such that the corresponding multiplicities fulfill $m_1 \geq \ldots \geq m_n$. If we have a cluster where not all the points are infinintely near to a single point, we can still find such a representative with the help of [5], Lemma 2.6: this lemma allows to find an isomorphism $\Phi : \operatorname{Bl}_{p_n} \ldots \operatorname{Bl}_{p_1} \mathbb{P}^2 \leftarrow \operatorname{Bl}_{p'_n} \ldots \operatorname{Bl}_{p'_1} \mathbb{P}^2$, and a unique permutation σ_Φ of $\{1, \ldots, n\}$ such that $\Phi^*(\widehat{E_i}) = E_{\sigma_\Phi(i)}$, such that $Z = m_1 p_1 + \ldots + m_n p_n$ is equivalent to $Z' = m_{\sigma_\Phi(1)} p'_1 + \ldots + m_{\sigma_\Phi(n)} p'_n$ and $m_{\sigma_\Phi(1)} \geq \ldots \geq m_{\sigma_\Phi(n)}$. We will need this fact in the proof of Theorem 2.

EXAMPLE 1. 1. Let $p_1 \in \mathbb{P}^2$, $p_2 \in \operatorname{Bl}_{p_1} \mathbb{P}^2 \setminus E_1$, $p_3 \in \operatorname{Bl}_{p_2} \operatorname{Bl}_{p_1 \mathbb{P}^2} \setminus (E_1 \cup E_2)$ and so on. Let $\mathcal{K} = [(p_1, \ldots, p_n)]$ and $\underline{m} = (m_1, \ldots, m_n)$. That $(\mathcal{K}, \underline{m})$ satisfies the proximity relations means nothing else than $m_i \geq 0$ for all i. Then $I(\mathcal{K}, \underline{m}) = \mathfrak{m}_{p_1}^{m_1} \cap \ldots \cap \mathfrak{m}_{p_n}^{m_n}$ where $\mathfrak{m}_{p_i}$ denotes the vanishing ideal of $(\pi_{i-1} \circ \ldots \circ \pi_1)(p_i)$.

2. Let $p_1 = (0 : 0 : 1) \in \mathbb{P}^2$. In $\operatorname{Bl}_{p_1} \mathbb{P}^2$, let p_2 be the point in E_1, corresponding to the line $\{x = 0\}$ in $\mathbb{P}^2$. Let p_3 be the intersection point of $\widehat{E_1}$ and E_2 in $\operatorname{Bl}_{p_2} \operatorname{Bl}_{p_1} \mathbb{P}^2$. Then $([(p_1, p_2, p_3)], (2, 1, 1))$ is a cluster satisfying the proximity relations and the corresponding fat point ideal is $I = \langle x^2 z, xy^2, y^3 \rangle \subset K[x, y, z]$. This is the cluster that appears if we blow up the singularity of the cusp $\{x^2 z - y^3 = 0\}$.

REMARK 1. An equivalent definition for the proximity relations is the following: A cluster $(\mathcal{K} = [(p_1, \ldots, p_n)], \underline{m})$ satisfies the proximity relations if for all strict transforms $\widehat{E_k}$ of exceptional divisors on the blown up surface X,

$$(-m_1 E_1 - \ldots - m_n E_n).\widehat{E_k} \geq 0 \text{ for all } k.$$

This formulation is equivalent to the above, as $\widehat{E_k} = E_k - \sum_{p_j \exists\exists \mathcal{K}\, p_k} E_j$.

See [2] for more explanations about clusters.

3. Changing the problem

We want to transform the problem of finding the free resolution of a fat point ideal step by step into a different problem. Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal, that is, $(\mathcal{K}, \underline{m})$ is a non empty cluster satisfying the proximity relations. Choose a representative $(p_1, \dots, p_n)$ in $\mathcal{K}$ such that $m_1 \geq \dots \geq m_n$. As before, let $X = \mathrm{Bl}_{p_n} \dots \mathrm{Bl}_{p_1} \mathbb{P}^2$. As codim $I = 2$ we can use the Auslander-Buchsbaum-Formula to compute the length of the free resolution of I. The free resolution of I has length 1, i.e. it has the following form:

$$0 \to M_1 = \bigoplus_{d \geq 0} R[-d]^{s_d} \to M_0 = \bigoplus_{d \geq 0} R[-d]^{v_d} \to I \to 0.$$

3.1. $v_d = \dim_K \operatorname{coker} \mu_d$ **and** $s_d = v_d - h_I(d) + 3h_I(d-1) - 3h_I(d-2) + h_I(d-3)$

v_d is the number of generators in degree d in a minimal generation system of I. Denote by μ_d the *multiplication map*

$$\mu_d : I_{d-1} \otimes R_1 \longrightarrow I_d.$$

Then $v_d = \dim_K \operatorname{coker} \mu_d$, as the image of μ_d contains all those forms of degree d in I that are a multiple of some form of lower degree in I. The equation for the s_d, $s_d = v_d - h_I(d) + 3h_I(d-1) - 3h_I(d-2) + h_I(d-3)$, can be seen if we tensor the free resolution with K and consider the Koszul complex of K.

3.2. $\dim \operatorname{coker} \mu_d = \dim_K \operatorname{coker} \mu_{X,F,E_0}$ **for a special linear system** F **on the blown up surface** X **and** $h_I(d) = h^0(X, \mathcal{O}_X(F))$

As already mentioned in the introduction, $h_I(d) = h^0(X, \mathcal{O}_X(dE_0 - m_1E_1 - \dots - m_nE_n))$. This follows from the fact that $I = \pi_*(\mathcal{O}_X(-m_1E_1 - \dots - m_nE_n)$ (where $\pi : X \to \mathbb{P}^2$ is the map of all blow ups) which can be seen considering local coordinates. Let F_d denote the linear system $F_d = dE_0 - m_1E_1 - \dots - m_nE_n$ on X. Then $h_I(d) = h^0(X, \mathcal{O}_X(F_d))$. Also, the multiplication map $\mu_d : I_{d-1} \otimes R_1 \longrightarrow I_d$ corresponds to a map

$$\mu_{X,F_d,E_0} : H^0(X, \mathcal{O}_X(F_{d-1})) \otimes H^0(X, \mathcal{O}_X(E_0)) \longrightarrow H^0(X, \mathcal{O}_X(F_d)).$$

3.3. $h^0(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(H))$ **if** $F = H + N$ **where** F **is fixed and** H **is nef and** $\dim_K \operatorname{coker} \mu_{X,F,E_0} = \dim_K \operatorname{coker} \mu_{X,H,E_0} + h^0(X, \mathcal{O}_X(F + E_0)) - h^0(X, \mathcal{O}_X(H + E_0))$

If we have a decomposition of F such that $F = H + N$ where N is fixed in F and H is nef, then $h^0(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(H))$. Furthermore, by the identification

$H^0(X, \mathcal{O}_X(H)) = H^0(X, \mathcal{O}_X(F))$ the image of

$$H^0(X, \mathcal{O}_X(H)) \otimes H^0(X, \mathcal{O}_X(E_0)) \longrightarrow H^0(X, \mathcal{O}_X(H + E_0)) \longrightarrow$$
$$\longrightarrow H^0(X, \mathcal{O}_X(F + E_0))$$

is equal to the image of μ_{X,F,E_0}, and hence we get the relation:

$$\dim_K \operatorname{coker} \mu_{X,F,E_0} = \dim_K \operatorname{coker} \mu_{X,H,E_0} +$$
$$+ h^0(X, \mathcal{O}_X(F + E_0)) - h^0(X, \mathcal{O}_X(H + E_0)).$$

So finally, if we can find

1. a decomposition in nef part H and fixed part N for a given linear system on X,
2. $h^0(X, \mathcal{O}_X(H))$, and
3. $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ for a nef linear system H,

we can give the free resolution of the ideal I. However, it is not possible to solve these problems in general. In the present paper we investigate them in the situation where the points lie on a conic. The solution will be presented in the following section. For a more detailed account on the facts mentioned in this section, see [6], Chapters 1 and 2.

4. Points on a conic

Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal, and let Q' be a conic in $\mathbb{P}^2$. Let $(p_1, \ldots, p_n) \in \mathcal{K}$ such that $m_1 \geq \ldots \geq m_n$ and assume that for all i, p_i lies on the strict transform of Q' in $\mathrm{Bl}_{p_{i-1}} \ldots \mathrm{Bl}_{p_1} \mathbb{P}^2$. Denote by Q the stric transform of Q' in X. Then the linear system of Q is $2E_0 - E_1 - \ldots - E_n$ and so the latter is non empty. Furthermore, the linear system of $E_0 + Q = -K_X$ is non empty. These two non empty linear systems are important to get solutions for our three problems.

4.1. The decomposition

The first problem is to find a decomposition of a linear system F in nef part and fixed part. Assume that $F \neq \emptyset$. The idea is, if we have a finite number of irreducible curves with negative self-intersection on X, we can just intersect F with those curves and subtract those which have negative intersection with F. The remaining system is nef. In the case that the points lie on a conic, the number of curves of negative self-intersection is finite. More precisely, the only curves of negative self-intersection are components of E_i for some $i > 0$, components of Q or strict transforms of lines through the points. The idea to prove this is to assume that a curve C of negative self-intersection would fulfill $C.E_0 > 2$. Then $-C.K_X = C.(Q + E_0) \geq C.E_0 > 2$ as C cannot be a component of Q, and we get a contradiction to negative self-intersection by the adjunction formula, as $p_a(C) \geq 0$. For a detailed proof, see [5], Lemma III.i.1.(a).

4.2. $h^0(X, \mathcal{O}_X(H))$ for a nef linear system H

The second problem is to compute $h^0(X, \mathcal{O}_X(H))$ for a nef linear system H on X. [4], Lemma II.2(c), tells us that $h^2(X, \mathcal{O}_X(H)) = 0$ for a nef linear system H on any surface X with $-K_X \neq \emptyset$, and [5], Lemma III.I.1 (b), tells us that $h^1(X, \mathcal{O}_X(H)) = 0$ for a nef linear system H on X, where the blown up points lie on a conic. The same lemma tells us that H has no fixed components, that is if we have a decomposition of a linear system F in part N which is fixed in F and a part H which is nef, then N is the fixed part of F and H has no fixed part. So by Riemann-Roch we get

$$h^0(X, \mathcal{O}_X(H)) = (H^2 - K_X.H)/2 + 1.$$

4.3. $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ for a nef linear system H

The third problem is to compute $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ for a nef linear system H on X. Harbourne gives a solution for this problem, too, but as one Lemma ([5], Lemma II.5) which he uses in his proof seems not to hold in the generality stated there, we will give a correct version of his proof here. We need a lemma in advance:

LEMMA 1. *Let N be a reduced divisor on X such that $|N| \neq \emptyset$ and $h^0(X, \mathcal{O}_X(N + K_X)) = 0$. Assume N has at most two components. Let F and G be divisors on X meeting every component of N nonnegatively. If N has only one component, then the multiplication map $\mu_{N,F,G} : H^0(N, \mathcal{O}_N(F)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(N, \mathcal{O}_N(F + G))$ fulfills* coker $\mu_{N,F,G} = 0$. *If N has two components, then we get the same result if we have $G.C_2 \geq C_1.C_2$, $G.C_1 \geq C_2.C_1$ and $F.C_1 \geq C_2.C_1$, where C_1 and C_2 are the two components of N.*

Proof. As $0 = H^0(X, \mathcal{O}_X(N+K_X)) = H^2(X, \mathcal{O}_X(-N))$ by assumption, we see from the cohomology sequence of the structure sequence of N in X that $h^1(N, \mathcal{O}_N) = 0$. Then by [1], Theorem 1.7, it follows that the components of N are rational. Assume first that $N \cong \mathbb{P}^1$ has only one component. As F and G meet N non negatively, we get effective divisors on N and the map $\mu_{N,F,G} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(G)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F + G))$ maps a tensor product of two polynomials of degree $F.N$ respectively $G.N$ in two variables to their product. This map is surjective, as $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F+G))$ is generated by the monomials in degree $F.N+G.N$ and we can give a preimage for every such monomial easily. So $\mu_{N,F,G}$ is surjective and $\dim_K \operatorname{coker} \mu_{N,F,G} = 0$. Now assume that N has two components C_1 and C_2. By the same argument as above, we see $\operatorname{coker} \mu_{C_2,F,G} = 0$ and $\operatorname{coker} \mu_{C_1,F-C_2,G} = 0$ since by assumption $(F - C_2).C_1 \geq 0$. Consider the sequence

$$0 \to \mathcal{O}_{C_1}(-C_2) \to \mathcal{O}_N \to \mathcal{O}_{C_2} \to 0.$$

From this we can get the following diagram, after tensoring the sequence with F, respectively the corresponding long cohomology sequence with $H^0(N, \mathcal{O}_N(G))$ (in the diagram, we use $H^0(N,G)$ as a shortcut for $H^0(N,\mathcal{O}_N(G))$ and analogous shortcuts):

$$\begin{array}{ccccccccc}
0 \to & \begin{array}{c} H^0(C_1, F-C_2) \\ \otimes \\ H^0(N,G) \end{array} & \to & \begin{array}{c} H^0(N,F) \\ \otimes \\ H^0(N,G) \end{array} & \to & \begin{array}{c} H^0(C_2,F) \\ \otimes \\ H^0(N,G) \end{array} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(C_1, F+G-C_2) & \to & H^0(N, F+G) & \to & H^0(C_2, F+G) & \to 0
\end{array}$$

where the second vertical map is $\mu_{N,F,G}$.

The rows are exact, because $h^1(C_1, F-C_2) = h^1(C_1, F+G-C_2) = 0$ (by Serre duality, and since C_1 is rational and as $(F-C_2).C_1 \geq 0$ by assumption) and since $H^0(N,G)$ is free. From this diagram we get with the Snake Lemma an exact sequence

$$\operatorname{coker}\big(H^0(C_1, F-C_2) \otimes H^0(N,G) \to H^0(C_1, F+G-C_2)\big) \longrightarrow \operatorname{coker}\mu_{N,F,G}$$

$$\longrightarrow \operatorname{coker}\big(H^0(C_2, F) \otimes H^0(N,G) \to H^0(C_2, (F+G))\big) \to 0.$$

As $h^1(C_1, \mathcal{O}_{C_1}(G-C_2)) = 0$ (again by Serre duality, and since C_1 is rational and $(G-C_2).C_1 \geq 0$) we can see from the sequence above that $H^0(N,\mathcal{O}_N(G)) \to H^0(C_2, \mathcal{O}_{C_2}(G))$ is surjective and so

$$\operatorname{coker}\big(H^0(C_2, \mathcal{O}_{C_2}(F)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(C_2, \mathcal{O}_{C_2}(F+G))\big) =$$

$$= \operatorname{coker}\mu_{C_2,F,G} = 0.$$

Reversing the role of C_1 and C_2 in the sequence we also get $H^0(N,\mathcal{O}_N(G)) \to H^0(C_1, \mathcal{O}_{C_1}(G))$ is surjective (as $h^1(C_2, \mathcal{O}_{C_2}(G-C_1)) = 0$ as $(G-C_1).C_2 \geq 0$) and hence

$$\operatorname{coker}\big(H^0(C_1, \mathcal{O}_{C_1}(F-C_2)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(C_1, \mathcal{O}_{C_1}(F+G-C_2))\big)$$
$$= \operatorname{coker}\mu_{C_1,F-C_2,G} = 0.$$

But then of course $\operatorname{coker}\mu_{N,F,G} = 0$. □

LEMMA 2. *Let $X = Bl_{p_n} \dots Bl_{p_1}\mathbb{P}^2$ such that $p_1, \dots, p_n$ lie on a conic. Let H be a nef linear system on X. Then* $\dim_K \operatorname{coker}\mu_{X,H,E_0} = 0$.

Proof. We know that we can write $H = a_0E_0 - a_1E_1 - \dots - a_nE_n$. H nef implies $a_i \geq 0$ for all i. The proof will be an induction on a_0. Assume $a_0 = 0$. Then $a_i = 0$ for all i as otherwise $H = \emptyset$ which is not possible by [5], Lemma 3.1.1.(b). Assume $a_0 = 1$. Then only one a_i can be non zero, as otherwise H would not be nef. So without restriction $H = E_0 - E_1$. Consider the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_H(H) \to 0.$$

Taking the long cohomology sequence and tensoring it with $H^0(X, \mathcal{O}_X(E_0))$, we can again see with the Snake Lemma that $\operatorname{coker}\mu_{X,H,E_0} = \operatorname{coker}\mu_{H,H,E_0}$. But the latter is zero by Lemma 1, as H has only one irreducible component, H and E_0 intersect this component non negatively and $H^0(X, \mathcal{O}_X(H + K_X)) = 0$. So assume $a_0 \geq 2$. Then we can subtract the linear system of the conic Q (which we will by abuse of notation also call Q) from H, and we still get a nef linear system, which can be tested intersecting $H - Q$ with the possible curves of negative self-intersection (see [5], Lemma III.i.1.(c)). So the induction assumption tells us that $\dim_K \operatorname{coker} \mu_{X,H-Q,E_0} = 0$. With Ramanujan-vanishing (see [7], Theorem 1) we can see that $h^1(X, \mathcal{O}_X(H - Q)) = 0$ and $h^1(X, \mathcal{O}_X(E_0 - Q)) = 0$. The first gives us an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(H - Q)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(Q, \mathcal{O}_Q(H)) \to 0$$

which we can tensor with $H^0(X, E_0)$ and apply Snake Lemma to get an exact cokernel sequence

$$\begin{aligned} &\operatorname{coker} \mu_{X,H-Q,E_0} \to \operatorname{coker} \mu_{X,H,E_0} \to \\ &\qquad \operatorname{coker}\left(H^0(Q, \mathcal{O}_Q(H)) \otimes H^0(X, \mathcal{O}_X(E_0)) \to H^0(Q, \mathcal{O}_Q(H + E_0))\right). \end{aligned}$$

The latter tells us that $H^0(X, \mathcal{O}_X(E_0)) \to H^0(Q, \mathcal{O}_Q(E_0))$ is surjective, and so the last cokernel in the sequence is just $\operatorname{coker} \mu_{Q,H,E_0}$. We want to see that $\dim_K \operatorname{coker} \mu_{Q,H,E_0}$ is also zero, as then $\operatorname{coker} \mu_{X,H,E_0} = 0$ with the induction assumption. To see this, apply Lemma 1 with $N = Q$, $F = H$ and $G = E_0$. As $Q = 2E_0 - E_1 - \ldots - E_n$, $h^0(X, \mathcal{O}_X(Q + K_X)) = 0$. There are two cases: either Q has only one component, or Q is a union of two lines C_1 and C_2. These lines can intersect at most with multiplicity 1, so $H.C_1 \geq C_1.C_2 = 1$ and $1 = E_0.C_2 = E_0.C_1 = C_1.C_2$. In any case, the assumptions of Lemma 1 are fulfilled and we get $\operatorname{coker} \mu_{Q,H,E_0} = 0$ and hence with the cokernel sequence from above, $\operatorname{coker} \mu_{X,H,E_0} = 0$. □

5. The proof of theorem 2

With these results, we are now able to compute a free resolution for points that lie on a conic. The Theorems 1 and 2 give a free resolution explicitly for two cases: if all but one of the points lie on a line and the point which is not on the line has multiplicity 1, and if all points lie on a smooth conic and the highest multiplicity occurs four times. The proofs for both theorems are quite similar, using the results from above. Here, we just want to give a proof for Theorem 2.

GENERAL REQUIREMENT 1. Let $I = I(\mathcal{K}, \underline{v})$ be a fat point ideal, that is, $(\mathcal{K}, \underline{v})$ is a nonempty cluster satisfying the proximity relations. Assume that the highest multiplicity, m, i.e. the highest entry in the vector $\underline{v}$, occurs four times in $\underline{v}$. Choose a representative $(p, q, r, s, p_1, \ldots, p_n)$ in $\mathcal{K}$ such that the highest multiplicity m is associated to p, q, r and s and such that for the other associated multiplicities, $m_1 \geq \ldots \geq m_n$ holds. Assume that p, q, r, s and $p_1, \ldots, p_n$ lie on a conic, respectively on the strict transform of the conic. Let $X = \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1} \mathrm{Bl}_s \mathrm{Bl}_r \mathrm{Bl}_q \mathrm{Bl}_p \mathbb{P}^2$ and let Q denote the strict transform of the conic in X.

The restriction that the multiplicity m occurs four times is necessary, as otherwise the computations would be too complicated. We want to calculate the free resolution of the ideal I. First we have to compute the smallest degree d such that $h_I(d) > 0$ - the corresponding linear systems are then non empty and we can compute their decomposition as described in 4.1. As the linear system of all conics passing through the four points p, q, r and s has dimension one, we can certainly find a conic $Q' \subset X$ different from Q that goes through the four points p, q, r and s. If these four points are all of level zero (i.e. they are not infinitely near to any other point), $\pi(Q')$ is a conic through the four points in $\mathbb{P}^2$. Then $\pi(Q')$ has multiplicity one in each of the four points. If some of the four points are infinitely near, $\pi(Q')$ goes through their images in $\mathbb{P}^2$, and the sum of the multiplicities in the image points is in any case bigger than or equal to four. Assume that $h_I(d) > 0$. Then with Bézout's Theorem (see [3], page 53) we know that

$$2d = \deg\big(\pi(F_d)\big) \cdot deg\big(\pi(Q')\big) \geq 4m$$

so $d \geq 2m$. But as $m \cdot Q \in I_{2m}$ we can see that $h_I(2m) > 0$ and so the smallest degree for which $h_I(d) > 0$ is $2m$. So

$$F_{2m} = 2mE_0 - mE_p - mE_q - mE_r - mE_s - m_1E_1 - \ldots - m_nE_n$$

is in a non empty linear system and we have to find curves in the fixed part of this linear system in order to calculate $h^0(X, \mathcal{O}_X(F_{2m}))$. From 4.1 we know that the only irreducible curves that we can find in the fixed part, are components of E_i for some $i \in \{p, q, r, s, 1 \ldots, n\}$, $E_0 - E_i - E_j$ where $i \neq j$ and $i, j \in \{p, q, r, s, 1, \ldots, n\}$, or the conic Q itself, which is irreducible. Let us try Q first.

$$F_{2m}.Q = 4m - m - m - m - m - m_1 - \ldots - m_n < 0$$

so Q is fixed, and after subtracting it, we get

$$\begin{aligned} F_{2m} - Q =& (2m-2)E_0 - (m-1)E_p - (m-1)E_q - (m-1)E_r - (m-1)E_s \\ & - (m_1 - 1)E_1 - \ldots - (m_n - 1)E_n. \end{aligned}$$

Intersecting this new linear system again with Q, we get

$$4m - 4 - 4(m-1) - \sum_{i=1}^{n}(m_i - 1)$$

which is negative if $m_1 \geq 2$. If this is the case, we can again subtract Q and we can see that going on like this, the first 4 points will "cancel" the positive part in the sum coming from $(F_{2m} - iQ).E_0 \cdot 2$, and the sum of the remaining n points counts negatively. So we can subtract Q altogether m_n times. But again, $(F_{2m} - m_n \cdot Q).Q < 0$, and we can subtract Q one more time, ending up with a linear system F'_{2m} such that $F'_{2m}.E_n = -1$. But then we can subtract E_n and have again a linear system with only negative multiplicities for the E_i, and because then the sum of the multiplicities of the $n - 1$ remaining points counts purely negatively again, we can as above subtract Q once more and so on. Finally, we end up with the linear system

$$H_{2m} = (2m - 2m_1)E_0 - (m - m_1)E_p - (m - m_1)E_q - (m - m_1)E_r - (m - m_1)E_s.$$

Now trying all irreducible curves of negative self-intersection, and seeing that they intersect H_{2m} non negatively, we find that H_{2m} is nef. Remark 1 tells us that we do not have to check the components of the E_i. But then we know by 4.2 that $h^1(X, \mathcal{O}_X(H_{2m})) = 0$ and

$$h^0(X, \mathcal{O}_X(H_{2m})) = \big((2m-2m_1+1)(2m-2m_1+2)-4(m-m_1)(m-m_1+1)\big)/2 =$$
$$= m - m_1 + 1.$$

Next we have to ask ourselves up to which degree d the curve $m_1 \cdot Q$ is fixed in F_d. Call $r_{m_i} = \sharp\{j \in \{1, \ldots, n\} \mid m_j = m_i\}$. If $r_{m_1} = 1$, then already $(F_{2m+1} - (m_1 - 2) \cdot Q).Q = 4m + 2 - 4m_1 + 8 - 4m + 4m_1 - 8 - 2 = 0$, so Q is not for sure fixed the $(m_1 - 1)-$st time in F_{2m+1}, as this intersection is not less than zero, and so only $(m_1 - 2) \cdot Q$ is in the fixed part of F_{2m+1}.

NOTATION 2. We use the following notation

$$\begin{aligned} F_d - t \cdot Q = (d-t)E_0 - \max(m-t,0)E_p - \ldots - \max(m-t,0)E_s \\ - \max(m_1 - t, 0)E_1 - \ldots - \max(m_n - t)E_n. \end{aligned}$$

The change of the fixed part described above happens because the four points with multiplicity m always "cancel" the "$2m-$part" of $F_{2m+a}.E_0$ - so if we enlarge the degree by one, we get two more in the E_0-part of the intersection $F_{2m+a}.Q$, and two less, each time we subtract Q, but we get only one less in the $E_1, \ldots, E_n-$part when we subtract Q for the last times, because then only E_1 is left. So for the first degrees, up to the degree where E_2 appears with a multiplicity bigger than zero in the fixed part of F_d, there is a difference in the change of the fixed parts: for every degree d F_d has Q two times less in its fixed part than F_{d-1}. Call $c := m_1 - m_2$. As long as $a \leq \lfloor c/2 \rfloor$, the fixed part of F_{2m+a} is $(m_1 - 2a) \cdot Q$, as

$$\begin{aligned} (F_{2m+a} - &(m_1 - 2a - 1)Q).Q \\ &= ((2m + a - 2(m_1 - 2a - 1))E_0 - (m - m_1 + 2a + 1)E_p \\ &\quad - \ldots - (m - m_1 + 2a + 1)E_s - (2a+1)E_1).Q \\ &= 4m + 2a - 4m_1 + 8a + 4 - 4m + 4m_1 - 8a - 4 - 2a - 1 = -1 < 0 \end{aligned}$$

and

$$\begin{aligned} (F_{2m+a} - &(m_1 - 2a)Q).Q \\ &= ((2m + a - 2(m_1 - 2a))E_0 - (m - m_1 + 2a)E_p \\ &\quad - \ldots - (m - m_1 + 2a)E_s - (2a)E_1).Q \\ &= 4m + 2a - 4m_1 + 8a - 4m + 4m_1 - 8a - 2a = 0. \end{aligned}$$

The next step now depends on whether c is even or odd, because in the first case, $(m_1 - c)Q$ is fixed in $F_{2m+c/2}$, and in the other case, $(m_1 - c + 1)Q$ is fixed in $F_{2m+(c-1)/2}$, and we have to decide about what is fixed in the following degrees with this knowledge. So we make a distinction on whether c is even or odd. Let us start with the case that c is even. As we have already seen, we get two times Q less in the fixed part for every degree from $2m$ up to $2m + c/2$. So for $0 \leq a \leq c/2$, the fixed part of F_{2m+a} is $(m_1 - 2a)Q$ and we can compute

$$\begin{aligned} h_I(2m+a) =& h^0\big(X, \mathcal{O}_X(F_{2m+a})\big) = h^0\big(X, \mathcal{O}_X(F_{2m+a} - (m_1 - 2a)Q)\big) \\ =& h^0\big(X, \mathcal{O}_X((2m + a - 2(m_1 - 2a))E_0 - (m - m_1 + 2a)E_p \\ & - \ldots - (m - m_1 + 2a)E_s - 2aE_1)\big) \\ =& \frac{1}{2}((2m + 5a - 2m_1 + 1)(2m + 5a - 2m_1 + 2) \\ & - 4(m - m_1 + 2a)(m - m_1 + 2a + 1) - 2a(2a + 1)). \end{aligned}$$

Now for the degrees following $2m + c/2$, it depends on how often the multiplicity m_2 appears, up to which degree $(m_1 - c)Q$ is in the fixed part. From the degree where Q is less than m_2 times in the fixed part, we have more than only one E_i left in the nef part of the linear system F_{2m+a}, and so in the intersection sum $H_{2m+a}.Q$, we get two less in the E_0- part when we subtract Q and two or more less in the $E_1, \ldots, E_n-$part, so it can no longer happen that the fixed components differ by $2Q$ for each degree. This means we can go on in the following way: we know that for the degrees following $2m + c/2$, m_2Q is in the fixed part, and we have to ask ourselves which is the degree for which that is no longer true, i.e. for which degree $2m + a$ only $(m_2 - 1)Q$ is in the fixed part. The inequality that tells us Q is fixed the m_2-th time is $(F_{2m+a} - (m_2 - 1)Q).Q < 0$, so the degree for which this is no longer true, is the degree for which

$$\begin{aligned} 0 &= (F_{2m+a} - (m_2 - 1)Q).Q \\ &= ((2m + a - 2m_2 + 2)E_0 - (m - m_2 + 1)E_p \\ &- \ldots - (m - m_2 + 1)E_s - (c + 1)E_1 - E_2 - \ldots - E_{\mu_{m_2}+1}).Q \\ &= 4m + 2a - 4m_2 + 4 - 4m + 4m_2 - 4 - c - 1 - \mu_{m_2} = 2a - c - 1 - \mu_{m_2} \end{aligned}$$

$$\Leftrightarrow 2a = c + 1 + \mu_{m_2} \tag{1}$$

where μ_{m_2} is defined with the help of the *Young diagram* of $(m_2, \ldots, m_n)$, this is a diagram consisting of $n - 1$ rows of boxes, the bottom row consisting of m_2 boxes, the second of m_3 boxes, and so on, and the top one consisting of m_n boxes. The *conjugate Young diagram* is the diagram $(\mu_1, \ldots, \mu_{m_2})$ where μ_i is the number of boxes in the i-th column of the Young diagram of $(m_2, \ldots, m_n)$. But the equation above might not be solvable as $c+1+\mu_{m_2}$ might be odd. Then we have to ask ourselves, for which value of a the intersection of $F_{2m+a} - (m_2 - 1)Q$ with Q is no longer negative. $2a$ is of course always even, and if $c+1+\mu_{m_2}$ is odd, then $(F_{2m+a}-(m_2-1)Q).Q = 2a-(c+1+\mu_{m_2})$ is odd. So it can never be zero if we enlarge a by steps of one, but it will be -1 and then 1 and the next step - and this is the step to stop. To sum up, F_{2m+a} has no longer m_2Q in its fixed component if $a = \lceil \frac{c+1+\mu_{m_2}}{2} \rceil$. So we know for all

$$c/2 \leq a < \lceil \frac{c + 1 + \mu_{m_2}}{2} \rceil, \text{ that}$$

$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a})) = h^0(X, \mathcal{O}_X(F_{2m+a} - m_2 Q))$$

and as $h^1(X, \mathcal{O}_X(F_{2m+a} - m_2 Q)) = 0$ since this linear system is nef, the latter can be calculated easily with Riemann-Roch.

DEFINITION 2. *We will denote a degree d such that F_{d-1} has $i \cdot Q$ as fixed component whereas F_d only has $(i-1)\cdot Q$ as fixed component a* special degree *of the ideal I.*

We now have to ask, up to which degree $(m_2 - 1)Q$ is in the fixed part, that is, we have to find special degrees. Let us enumerate not the special degrees (as they depend on whether $c+1+\mu_{m_2}$ or a similar sum is even or odd), but these sums that tell us how big $2a$ (as in the equation 1) may be - call $a_{m_2} = c + 1 + \mu_{m_2}$, the first special degree is then $2m + \lceil a_{m_2}/2 \rceil$. We find a_{m_2-i} if we ask us for which degree $2m+a$, $(m_2-i)Q$ is no longer in the fixed part, i.e. for which degree $(F_{2m+a} - (m_2 - i - 1)Q).Q$ is no longer negative.

$$\begin{aligned}
&(F_{2m+a} - (m_2 - i - 1)Q).Q \\
&\quad= ((2m + a - 2(m_2 - i - 1))E_0 - (m - m_2 + i + 1)E_p \\
&\qquad - \ldots - (m - m_2 + i + 1)E_s - (c + i + 1)E_1 - (i+1)E_2 \\
&\qquad - \ldots - (i+1)E_{\mu_{m_2}+1} - iE_{\mu_{m_2}+2} \\
&\qquad - \ldots - iE_{\mu_{m_2-1}+1} - \ldots - E_{\mu_{m_2-i-1}+2} - \ldots - E_{\mu_{m_2-i}+1}).Q \\
&\quad= 2a - (c + i + 1 + \mu_{m_2} + \ldots + \mu_{m_2-i})
\end{aligned}$$

and so $a_{m_2-i} = c + i + 1 + \mu_{m_2} + \ldots + \mu_{m_2-i}$. So we know that for each

$$\left\lceil \frac{a_{m_2-i+1}}{2} \right\rceil \le a < \left\lceil \frac{a_{m_2-i}}{2} \right\rceil,$$

$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a})) = h^0(X, \mathcal{O}_X(F_{2m+a} - (m_2 - i)Q))$$

where $a_{m_2+1} := c$ for completeness. With these results, we are able to compute the Hilbert function for each degree in the case that c is even. The case that c is odd is done analogously, the result is:

1. for $0 \le a \le \frac{c-1}{2}$
$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a})) = h^0(X, \mathcal{O}_X(F_{2m+a} - (m_1 - 2a)Q));$$
2. for $\frac{c+1}{2} \le a < \left\lceil \frac{a_{m_2}}{2} \right\rceil$
$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a} - m_2 Q));$$
3. for $\left\lceil \frac{a_{m_2-i}}{2} \right\rceil \le a < \left\lceil \frac{a_{m_2-i-1}}{2} \right\rceil$
$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a} - (m_2 - i - 1)Q)).$$

Now we have to compute the v_d, the graded Betti numbers of the first module of the free resolution.

LEMMA 3. *Under the given assumptions, $v_d = 0$ for all degrees d which are not special and not equal to $m+1$.*

Proof. From 3.3 and since $\dim \operatorname{coker} \mu_{X,H,E_0} = 0$ for the nef part H of the linear system F_{d-1} by 4.3, we know that

$$v_d = \dim \operatorname{coker} \mu_{X,F_{d-1},E_0} = h^0(X, \mathcal{O}_X(F_{d-1}+E_0)) - h^0(X, \mathcal{O}_X(H+E_0)).$$

But we compute $h^0(X, \mathcal{O}_X(F_{d-1}+E_0)) = h^0(X, \mathcal{O}_X(F_d))$ as above: we find the fixed part of F_d and compute h^0 of the remaining nef part. As d is not a special degree, we know both F_{d-1} and F_d have the same fixed part $i \cdot Q$. But then

$$\begin{aligned} v_d &= h^0(X, \mathcal{O}_X(F_d)) - h^0(X, \mathcal{O}_X((F_{d-1} - i\cdot Q) + E_0)) \\ &= h^0(X, \mathcal{O}_X(F_d - i\cdot Q)) - h^0(X, \mathcal{O}_X(F_d - i\cdot Q)) = 0. \end{aligned}$$

□

So we only have to consider the cases where the fixed parts differ. This is for example the case for the degrees less than $c/2$, whether c is even or not, we know that for $a \le c/2$ respectively $a \le (c-1)/2$, F_{2m+a} has $(m_1 - 2a)Q$ as fixed part. So for $a \le c/2$ respectively $a \le (c-1)/2$,

$$v_{2m+a} = h^0(X, \mathcal{O}_X(F_{2m+a})) - h^0(X, \mathcal{O}_X(H_{2m+a-1} + E_0))$$

where H_{2m+a-1} is the nef part of F_{2m+a-1}

$$\begin{aligned} &= h^0(F_{2m+a} - (m_1 - 2a)Q)) - h^0(F_{2m+a-1} - (m_1 - 2(a-1))Q + E_0) \\ &= h^0((2m+5a-2m_1)E_0 - (m - m_1 + 2a)E_p - \\ &\quad - \ldots - (m - m_1 + 2a)E_s - 2aE_1) \\ &\quad - h^0((2m+5a-2m_1-4)E_0 - (m - m_1 + 2a - 2)E_p \\ &\quad - \ldots - (m - m_1 + 2a - 2)E_s - (2a-2)E_1) = 3 \end{aligned}$$

where we write $h^0(F_d)$ as a shortcut for $h^0(X, \mathcal{O}_X(F_d))$. In the case that c is odd, we know that also the fixed parts of $F_{2m+(c-1)/2}$ and $F_{2m+(c+1)/2}$ differ, so we have to compute $v_{2m+(c+1)/2}$. An analogous computation as above shows that $v_{2m+(c+1)/2} = 2$. At last, we have to compute v_d for the special degrees:

LEMMA 4. *For all special degrees $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$, $v_d = 1$ if a_{m_2-i} is even, and $v_d = 2$ if a_{m_2-i} is odd.*

Proof. Let $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$. Then F_d has the fixed part $(m_2 - i - 1)\cdot Q$ whereas

F_{d-1} has the fixed part $(m_2 - i) \cdot Q$. So

$$\begin{aligned}
h^0(F_d)-&h^0(H_{d-1}+E_0)\\
&= h^0(F_d - (m_2 - i - 1)\cdot Q) - h^0((F_{d-1} - (m_2 - i)\cdot Q) + E_0)\\
&= h^0((2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i + 2)E_0 - (m - m_2 + i + 1)E_p - \ldots\\
&-(m - m_2 + i + 1)E_s - (c + i + 1)E_1 - (i + 1)E_2 - \ldots\\
&-(i + 1)E_{\mu_{m_2}+1} - \ldots - E_{\mu_{m_2-i}+1})\\
&-h^0((2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i)E_0 - (m - m_2 + i)E_p - \ldots\\
&-(m - m_2 + i)E_s - (c + i)E_1 - iE_2 - \ldots\\
&-iE_{\mu_{m_2}+1} - \ldots - E_{\mu_{m_2-i-1}+1})\\
&= \frac{1}{2}((2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i + 3)(2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i + 4)\\
&-4(m - m_2 + i + 1)(m - m_2 + i + 2) - (c + i + 1)(c + i + 2)\\
&-\mu_{m_2}(i + 1)(i + 2) - (\mu_{m_2-1} - \mu_{m_2})i(i + 1)-\\
&- \ldots - (\mu_{m_2-i} - \mu_{m_2-i-1})\cdot 1\cdot 2)\\
&-\frac{1}{2}(2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i + 1)(2m + \lceil a_{m_2-i}/2\rceil - 2m_2 + 2i + 2)\\
&-4(m - m_2 + i)(m - m_2 + i + 1) - (c + i)(c + i + 1) - \mu_{m_2}\cdot i\cdot(i + 1)\\
&\quad - \ldots - (\mu_{m_2-i-1} - \mu_{m_2-i-1})\cdot 1\cdot 2)\\
&= 2\cdot\lceil a_{m_2-i}/2\rceil + 1 - c - i - 1 - (i + 1)\mu_{m_2} - i\cdot\sharp\{i|m_i = m_2 - 1\}\\
&\quad - \ldots - \sharp\{i|m_i = m_2 - i\}\\
&= 2\cdot\lceil a_{m_2-i}/2\rceil + 1 - c - i - 1 - \mu_{m_2} - \ldots - \mu_{m_2-i}\\
&= 2\cdot\lceil a_{m_2-i}/2\rceil + 1 - a_{m_2-i}
\end{aligned}$$

and the latter is 1 if a_{m_2-i} is even and 2 if a_{m_2-i} is odd. □

For the first degree for which $h_I(d)$ is nonzero, we have $v_{2m} = h_I(2m) = m - m_1 + 1$. The calculation of the s_d, the Betti numbers of the second module in the free resolution, turns out to be much more complicated. As it can be calculated easily, $s_d = 0$ as long as the linear systems F_d, F_{d-1}, F_{d-2} and F_{d-3} have the same fixed part and $v_d = 0$. The cases for which $v_d \neq 0$ are easy to check. But the cases where the fixed parts of the linear systems vary are much more numerous, because for the degrees less than $c/2$ respectively $(c+1)/2$, if c is odd, the fixed part changes from degree to degree by $2Q$, and for bigger d, the fixed part changes at the special degrees. Here, we just want to give one example for a calculation of the s_d, for a complete investigation, see [6]. So let d be a special degree, $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$ for some i. Then v_d depends on whether a_{m_2-i} is even or odd, but we will just insert v_d in the calculation and insert the correct value for the two cases later. Assume $H_{2m+\lceil \frac{a_{m_2-i}}{2} \rceil} = d'E_0 - m'E_p - \ldots - m'E_s - m'_1E_1 - \ldots - m'_nE_n$, then $d' = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil - 2m_2 + 2i + 2$

and $4m' + m'_1 + \ldots + m'_n = 4m - 4m_2 + 4i + 4 + a_{m_2-i}$.

$$
\begin{aligned}
s_d =& v_d - \frac{1}{2}((d'+1)(d'+2) - 4m'(m'+1) - \sum_{i=0}^{n} m'_i(m'_i+1) \\
&+ \frac{3}{2}((d'-2)(d'-1) - 4(m'-1)m' - \sum_{i=1}^{n}(m'_i-1)m'_i) \\
&- \frac{3}{2}((d'-3)(d'-2) - 4(m'-1)m' - \sum_{i=1}^{n}(m'_i-1)m'_i) \\
&+ \frac{1}{2}((d'-4)(d'-3) - 4(m'-1)m' - \sum_{i=1}^{n}(m'_i-1)m'_i) \\
=& v_d + \frac{1}{2}(-3d' - 9d' + 15d' - 7d' - 2 + 6 - 18 + 12 + 4m' \\
&+ \sum_{i=1}^{n} m'_i + 3(4m' + \sum_{i=1}^{n} m'_i) - 3(4m' + \sum_{i=1}^{n} m'_i) + 4m' + \sum_{i=1}^{n} m'_i) \\
=& v_d - 2d' - 1 + 4m' + \sum_{i=1}^{n} m'_i \\
=& v_d - 4m - 2\lceil \frac{a_{m_2-i}}{2} \rceil + 4m_2 - 4i - 4 - 1 + 4m - 4m_2 + 4i + 4 + a_{m_2-i} \\
=& v_d - 2\lceil \frac{a_{m_2-i}}{2} \rceil - 1 + a_{m_2-i} = 0,
\end{aligned}
$$

in any of the two cases, because if a_{m_2-i} is even, then $2 \cdot \lceil \frac{a_{m_2-i}}{2} \rceil = a_{m_2-i}$ and $v_d = 1$, and if a_{m_2-i} is odd, then $2 \cdot \lceil \frac{a_{m_2-i}}{2} \rceil = a_{m_2-i} + 1$ and $v_d = 2$. Combining all the results, we finally get Theorem 2.

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QUERIES ON FAMILIES OF FAT POINTS

Abstract. Here we raise some questions on the postulation of non-generic unions of fat points in $\mathbf{P}^n$, e.g. for a fixed integer $z > 0, t > 0$ the dimension of all such Z's with $h^i(\mathbf{P}^n, \mathcal{I}_Z(t)) \geq z$, $i = 0, 1$.

Let X be an integral n-dimensional projective variety. Fix positive integers k and m_i, $1 \leq i \leq k$, and set $z = z(n; m_1, \ldots, m_k) := \sum_{i=1}^{k} \binom{n+m_i-1}{n}$. Let $A(X; m_1, \ldots, m_k)$ be the set of all zero-dimensional subschemes of X of the form $\cup_{i=1}^{k} m_i P_i$, where $P_1, \ldots, P_k$ are k distinct points of X_{reg}. Let $B(X; m_1, \ldots, m_k)$ be the closure of $A(X; m_1, \ldots, m_k)$ in the Hilbert scheme $\mathrm{Hilb}^z(X)$ of all length z zero-dimensional subschemes of X. Since $\mathrm{Hilb}^z(X)$ is a projective scheme, the variety $B(X; m_1, \ldots, m_k)$ is complete.

QUESTION 1. Fix $L \in \mathrm{Pic}(X)$ and set $\alpha_i := h^i(X, \mathcal{I}_Z \otimes L)$, $i = 0, 1$, where Z is the general element of $A(X; m_1, \ldots, m_k)$.

(a) Find upper and lower bounds for the dimension of integral subvarieties T of $A(X; m_1, \ldots, m_k)$ such that $h^i(X, \mathcal{I}_W \otimes L) > \alpha_i$ for every $W \in T$; more generally, fix an integer $a > 0$ and find bounds for $\dim(T)$ such that $h^i(X, \mathcal{I}_W \otimes L) \geq \alpha_i + a$ for every $W \in T$. More generally, do the same simultaneously for finitely many line bundles L.

(b) Find closed subvarieties $\bar{T}$ (if possible with large dimension) of $B(X; m_1, \ldots, m_k)$ which intersect $A(X; m_1, \ldots, m_k)$ and such that $h^i(X, \mathcal{I}_A \otimes L) = \alpha_i$ for every $A \in \bar{T}$ (or such that $h^i(X, \mathcal{I}_A \otimes L) \leq \alpha_i + a$ for every $A \in \bar{T}$).

(c) As in part (a) or (b) do the same taking a vector bundle E instead of L.

Part (b) means to find families of finite unions of fat points such that all their limits have good postulation or such that we may control the postulation of all their limits. For part (c) it is essential to consider only "important " or "nice " examples, e.g. sufficiently general stable vector bundles. When $X = \mathbf{P}^n$ part (c) for the bundles $\Omega^i_{\mathbf{P}^n}(t)$ is important for the minimal free resolution of finite unions of fat points in $\mathbf{P}^n$ and of their limits.

We are working on these questions, but our preliminary results do not kill the topic. For the case $X = \mathbf{P}^n$, $m_1 = \cdots = m_k = 1$ and $L = \mathcal{O}_{\mathbf{P}^2}(t)$, see [1].

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WHITE SURFACES AND THEIR TRISECANT LINES

Abstract. This note deals with the number n of trisecant lines passing through a generic point of a White surface S. Either $n = 6$ or $n = \infty^1$ and S is a Segre polygonal surface.

Let X be a surface in $\mathbb{P}^5$ with isolated singularities. Consider the projection π_l from a generic line l in $\mathbb{P}^5$, and denote by B the image of the singular locus of X. The image of X by π_l is a surface S in $\mathbb{P}^3$, such that $X \setminus B$ has a 1-dimensional locus of double points and a finite number of triple points. Since l is chosen generically, the fiber of π_l over a triple point p of $S \setminus B$ consists of 3 distinct points p_1, p_2, p_3.

What is their possible postulation, that is to say what is the dimension of their linear span?

Modern interest for questions of this sort stems out of the work of Pink- ham and Lazarsfeld ([10],[9])which proves the Castelnuovo-Mumford regularity conjecture for smooth surfaces. One could expects that for a generic choice of l this span $< p_1, p_2, p_3 >$ has dimension 3. This can be proved to be equivalent to say that the dimension of the trisecant lines locus of S is at most 3-dimensional. The existence of surfaces X with a 4-dimensional trisecant line locus therefore provide counter-example to this naive belief. We will say that such a surface possess an excess of trisecant lines. The only smooth example known is the Ferrara surface [5], that is to say a special Enriques surface of degree 10 in $\mathbb{P}^5$. Moreover Conte and Verra [3] have shown that this characterizes speciality for Enriques surfaces embedded in $\mathbb{P}^5$ by their Fano embedding. Historically the first example of surface with an excesss of trisecant lines known is a singular surface of degree 10 constructed by Segre in 1924. Let C be a plane conic and $l_1, \ldots, l_6$ be six tangent lines to C general enough so they meet two by two in 15 points P. The linear system of quintics passing by those points P is regular and define a surface S_0 of degree 10 in $\mathbb{P}^5$, that we'll call Segre polygonal surface. This surface belongs to the family of White surfaces ([12],[6] and [7]), that is to say surfaces of P^5 image of $\mathbb{P}^2$ by the rational map ϕ associated to the linear system of quintics passing by a group P of 15 distinct points not lying on any quartic curve. White surfaces are surfaces of degree 10 with possibly isolated 4-fold singularities all image of lines joingning 5 base points. Remark also that polygonal surfaces, i.e. White surfaces for which P is the group of points two by two cut out by six lines, are projective degenerations of Enriques surfaces of $\mathbb{P}^5$. Dobler has shown in his thesis that Segre's polygonal surface is a degeneration of special Enriques surfaces and that it is the only polygonal surface with an excess of trisecant lines. It is therefore natural to ask if there exists other White surfaces with an excess of trisecant lines. To tackle this problem one should notice that a trisecant line passing by a generic point $\phi(q)$ of S corresponds to a pair of points Π for which the linear system of quintics passing by $P + \Pi$ is 1-irregular. Using ideas of Gambier [6], especially his explicit construction of pairs of points associated to a generic point one can show the following [1]

THEOREM 1. *Let S be a White surface in $\mathbb{P}^5$ and $\phi(q)$ a generic point on it.*

1. *There pass at least one trisecant line to S by $\phi(q)$.*
2. *The surface S has an excess of trisecant lines exactly if S is a Segre polygonal surface.*
3. *If S is not a Segre polygonal surface, there pass exactly six trisecant lines to S by $\phi(q)$*

Let us point out that (1) was shown by Dobler in case S is smooth and (2) was already proved in 1882 by H.Krey [8] for a generic White surface. As a corollary of (1) one can show that the generic point of the principal componant of the the Hilbert scheme of 18 points in $\mathbb{P}^2$ irregular in degree 5 corresponds to 18 points not lying on any curve of degree 4. This improves in this very particular case a result of M.A. Coppo [4].

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BEYOND WARING'S PROBLEM FOR FORMS: THE BINARY DECOMPOSITION

Abstract. The study of the decomposition of a form of degree d in $n+1$ variables as the sum of forms involving $r \leq n$ variables is introduced. What is known for the sums of powers case $r = 1$ is illustrated and new results are presented in the binary case $r = 2$.

1. Introduction

Throughout this note S will denote the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. Given $f \in S_d$, a homogeneous form of degree d, the well known Waring's problem for forms deals with the study of expressions like: $f = {l_1}^d + \ldots + {l_s}^d$, where $l_1, \ldots, l_s \in S_1$ are linear forms. Noticing that ${l_i}^d$ is nothing more than a form in the univariate polynomial ring $\mathbb{C}[l_i]$, it is immediate to consider the following generalization: given $r \leq n$ and $f \in S_d$, study the decompositions of f of the form

$$f = f_1 + \ldots + f_s \tag{1}$$

where $f_i \in \mathbb{C}[y_{i1}, \ldots, y_{ir}]_d \subset S_d$ and the y_{ij}'s are linear forms, for $i = 1, \ldots, s$. In intuitive terms, we can consider the f_i's as forms in $n+1$ variables "essentially" involving r variables and (1) as a decomposition of f as the sum of forms involving a smaller number of variables. For $r = 1$, the f_i's are just pure powers and (1) is a sums of powers decomposition of f. For $r = 2$, the f_i's are called *binary forms* and (1) is called a *binary decomposition* of f. Notice that, e.g., $x_0^d + (x_1 + \cdots + x_n)^d$ is a binary form, while $\sum_{i=0}^n x_i^d$ is not a binary form for $n > 1$.

In the $r = 1$ case, the sums of powers case, the decomposition (1) can be performed for any d when $n = 1$ (see [2]) and for any n when $d = 2$ (this is just the diagonalization of a symmetric matrix) or $d = 3$ (see [3]). But there are not known algorithms for $n > 1$ and $d > 3$. In the more general case of $r \geq 2$ we can only try to exploit what we know in the sums of powers case, e.g. noticing that the sum of two pure powers is a binary form, but there are not dedicated procedures to compute (1). These remarks motivate our interest in a quantitative study of (1) and in particular in the investigation of the number of summands s. With this in mind we introduce $s_{min}(n, d) = \min\{s :$ exists (1) for a generic f in $S_d\}$, where the genericity assumption is made to have a behavior tamed enough to be studied in some generality. We can get an approximation of s_{min} by a parameters count: take general $f, f_1, \ldots, f_s$, i.e. with variable coefficients, and require the number of parameters in the left-hand side of (1) to be less or equal than the number of parameters in the right-hand side. This procedure gives an inequality and solving it we get $s_{exp}(n, d)$, i.e. the number of summands we expect in a decomposition of type (1) of a generic form of degree d in

$n+1$ variables. We remark that s_{exp} can be determined explicitly. Comparing this estimate with the number of summands appearing in the decomposition of a generic form of degree d in $n+1$ variables, it is easy to realize that $s_{min}(n,d) \geq s_{exp}(n,d)$. This inequality motivates our basic question: *(Q1) for which couples* (n,d) *does the equality* $s_{min}(n,d) = s_{exp}(n,d)$ *hold?* Question *(Q1)* addresses the problem of determining the minimal number of summands needed for the decomposition of a generic form. But, of course, there are *special* forms which can be decomposed using fewer summands. To study these forms we introduce the locus $\Sigma_s(n,d) \subseteq \mathbb{P}S_d$ of forms of degree d in $n+1$ variables which can be decomposed as in (1) using s summands. Hence it is natural to consider another question: *(Q2) what is the dimension of* $\Sigma_s(n,d)$*?* Actually, a parameters count gives an expected value for $\dim \Sigma_s(n,d)$. In this sense, question *(Q2)* is a generalization of question *(Q1)* .

In this note we will recall some known facts about the sums of powers case, $r=1$, and we will illustrate what it is known in the binary case, $r=2$ (for more details and proofs see [4]).

REMARK 1. To be rigorous, (1) represents a family of decompositions: one for each value of r. Hence we should mention r in the definitions, e.g., of s_{min} and s_{exp}. As different values of r will never appear in the same argument, we decide to keep the notation as simple as possible avoiding to mention r explicitly.

2. The sums of powers decomposition

Some particular instances of the sums of powers decomposition were classically studied by Clebsch, Darboux, London, Sylvester, Terracini and others (see [4] and the references there). Particular attention was devoted to the investigation of *defective* couples: a couple (n,d) is said to be defective if $s_{min}(n,d) \neq s_{exp}(n,d)$, i.e. if the generic form of degree d in $n+1$ variables can not be written as the sum of the expected number of powers of linear forms. A straightforward computation shows that $(n,2)$ is defective for all n and it can be shown that other defective couples exist, namely $(n,d) = (2,4), (3,4), (4,4), (4,3)$ (see [5]). All these defective couples were known since the beginning of the last century, but it was quite difficult to prove that no other defective couples exist. Finally, this was done by Alexander and Hirschowitz in 1995 (see [1]):

THEOREM 1. *A generic form of degree* d *in* $n+1$ *variables is the sum of* $s_{exp}(n,d) = \lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$ *sums of powers of linear forms, unless* $(n,d) = (2,4), (3,4), (4,4), (4,3)$ *and* $(n,2)$ *for all* n.

This Theorem completely answer to question *(Q1)* . Actually, the result by Alexander and Hirschowitz also gives a complete answer to question *(Q2)* showing that the expected behavior is the right one with few exceptions. More precisely, they determine $\dim \Sigma_s(n,d)$ which turns out to be the expected one unless few exceptions.

3. The binary decomposition

We are not aware of classical attempts to study the binary decomposition of forms and we will briefly illustrate the main results contained in [4], namely we will show what it is known for $n = 2, 3$. As before, a couple (n, d) will be said to be defective if $s_{min}(n, d) \neq s_{exp}(n, d)$, i.e. if the generic form of degree d in $n + 1$ variables can not be written as the sum of the expected number of binary forms. In the three variable case, $n = 2$, it is possible to determine a formula for s_{min} and using it we can show:

THEOREM 2. *For the binary decomposition of forms in three variables ($n = 2$), the only not defective couples are obtained for $d = 2, 3, 4, 5, 6$ and 8.*

We stress the sharp contrast with the sums of powers case where there are only few defective couples. The previous Theorem settles question *(Q1)* for the binary decomposition of forms in three variables. Concerning question *(Q2)* we can only show that $\Sigma_2(2, d)$ has dimension 1 less than expected for $d > 3$. The four variables case, $n = 3$, seems to be more complex and there are not known defective couples (actually it is conjectured that defective couples do not exist). In this case, the only results concern question *(Q1)* :

THEOREM 3. *For the binary decomposition of forms in four variables ($n = 3$), there are not defective couples for $d \leq 5$.*

4. Final remarks

For the purpose of this note, it is convenient to collect some interesting facts in the following remarks.

The sums of power decomposition of forms in two variables ($r = 1, n = 1$) is quite easy and question *(Q1)* was already answered by Sylvester. The binary decomposition of forms in three variables ($r = 2, n = 2$) can be successfully studied and a complete answer to question *(Q1)* is given in Theorem 2, while the problem is still open when $n > 2$. These facts suggests that the bigger the gap $n - r$ the more difficult the problem. Moreover, it is reasonable to think that the study of decompositions of type (1) for $r = n$ could be successfully attempted.

Although quite algebraic in their presentation, the decompositions of type (1) have a deep geometric nature. In the case $r = 1$, questions *(Q1)* and *(Q2)* can be naturally expressed in terms of the higher secant varieties of the Veronese $V_{n,d} = \nu_d(\mathbb{P}^n)$, where ν_d is the d-uple embedding. For example, to answer question *(Q2)* one has to determine $\dim \mathrm{Sec}^s(V_{n,d})$, i.e. the dimension of the variety of s-secant $\mathbb{P}^{s-1}$ to the Veronese. To give the same kind of geometric interpretation in the case $r = 2$, one needs to introduce the variety of binary forms $X_{n,d}$ which parameterizes the binary forms of degree d in $n + 1$ variables.

The variety of binary forms $X_{n,d}$ can be easily constructed from the Veronese $V_{n,d}$: consider a rational normal curve of degree d, $C \subset V_{n,d}$, and take its linear span $\langle C \rangle$;

making C to vary and taking the union of the linear spaces $\langle C \rangle$ one obtains $X_{n,d}$. Theorems 2 and 3 are obtained by studying $\mathrm{Sec}^s(X_{n,d})$ in terms of the Veronese, using a sort of Terracini's Lemma. It would be interesting to generalize this procedure: given a variety Y construct a new variety Z by taking "distinguished" subvarieties of Y and the union of their linear spans (this is an attempt to generalize the notion of higher secant variety where the "distinguished" subvarieties are just finite sets of points). Is there any analogous of Terracini's Lemma relating the geometry of $\mathrm{Sec}^s(Z)$ to the geometry of Y?

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RECENT RESULTS ON LINEAR SYSTEMS ON GENERIC $K3$ SURFACES

Abstract. In this note we relate about the problem of evaluate the dimension of linear systems through fat points defined on generic $K3$ surfaces.

1. Introduction and statement of the problem

In what follows we assume that the ground field is algebraically closed of characteristic 0. With S we always denote a smooth projective *generic* $K3$ surface, i.e. Pic $(S) = \langle H \rangle$ and let $n = H^2$. Consider r points in general position on S, to each one of them associate a natural number m_i called the *multiplicity* of the point. We will denote by $\mathcal{L} = \mathcal{L}^n(d, m_1, \ldots, m_n)$ the linear system $|dH|$ through the r points with the given multiplicities. Define the *virtual dimension* of the system as $v(\mathcal{L}) = d^2n/2 + 1 - \sum m_i(m_i + 1)/2$ and its *expected dimension* by $e = \max\{v, -1\}$. Observe that $e \leq \dim(\mathcal{L})$ and that the inequality may be strict if the conditions imposed by the points are dependent. In this case we say that the system is *special*. By S' we will denote the blow-up of S along the r points, given two curves A, B on S, the intersection AB will be defined as the intersection of their strict transforms on S'. The problem of classifying special systems has been largely studied for linear systems on the plane [2, 6, 11] and more generally for systems on rational surfaces [7, 8]. The main conjecture on the structure of such systems has been formulated in [8]. In this note we report about some recent results in the case of generic $K3$ surfaces. In [3] the authors proved that on the projective plane this conjecture is equivalent to an older one given by Segre in [11]. The advantage of Segre conjecture is that it can be formulated in the same way on any surface. Starting from this idea we proved in [4] the equivalence of Conjecture 1 with Conjecture 2 on a generic $K3$ surface. An attempt to prove Conjecture 2 has been done in [5] by using a degeneration technique inspired by [1]. The main result, by using this technique, is Theorem 1 which relates the speciality of some linear systems through points of the same multiplicity with the speciality of systems through just one point.

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2. The equivalence of the two conjectures

As stated in the introduction we consider here an extension, to any surface, of Segre conjecture about special linear systems.

CONJECTURE 1. If $\mathcal{L}$ is non-empty and reduced linear system on a surface S, then it is non-special.

By Bertini second theorem, this conjecture tell us that if $\mathcal{L}$ is special, then there exists an irreducible curve C such that $2C \subseteq \mathrm{Bs}(\mathcal{L})$. This means that, if Conjecture 1 is true, then in order to give a classification of special systems on a surface we should be able to classify the type of the curve C. In the case of generic $K3$ surfaces we proved the equivalence of the preceding conjecture with the following (see [4]).

CONJECTURE 2. Let $\mathcal{L}$ and S be as above, then

(i) $\mathcal{L}$ is special if and only if $\mathcal{L} = \mathcal{L}^4(d, 2d)$ or $\mathcal{L} = \mathcal{L}^2(d, d^2)$ with $d \geq 2$;

(ii) if $\mathcal{L}$ is non-empty then its general divisor has exactly the imposed multiplicities in the points p_i;

(iii) if $\mathcal{L}$ is non-special and has a fixed irreducible component C then

a) $\mathcal{L} = \mathcal{L}^2(m+1, m+1, m) = mC + \mathcal{L}^2(1, 1)$ with $C = \mathcal{L}^2(1, 1^2)$ or

b) $\mathcal{L} = 2C$ with $C \in \{\mathcal{L}^4(1, 1^3), \mathcal{L}^6(1, 2, 1), \mathcal{L}^{10}(1, 3)\}$ or

c) $\mathcal{L} = C$.

(iv) if $\mathcal{L}$ has no fixed components then either its general element is irreducible or $\mathcal{L} = \mathcal{L}^2(2, 2)$.

The proof of this result proceeds by analyzing the base locus of the system $\mathcal{L}$. Assume that there exist distinct irreducible curves C_i and D_j such that $\mathcal{L} = \sum \mu_i C_i + \sum D_i + \mathcal{M}$, where $\mu_i \geq 2$ and $\mathcal{M}$ has no fixed components. Given two irreducible curves $A, B \subseteq \mathrm{Bs}(\mathcal{L})$, by Conjecture 1 we have that $v(A) = v(B) = v(A+B) = 0$, but $v(A+B) = v(A) + v(B) + AB - 1$, so $AB = 1$. This gives $C_i C_j = C_i D_j = D_i D_j = 1$ and $C_i^2 \leq 1$. In (see [4]) we prove that given two irreducible curves A and B on S then either $AB \neq 1$ or $A = \mathcal{L}^2(1, 1^2)$ and B is an irreducible element of $\mathcal{L}^2(1, 1)$.

3. A degeneration of K3 surfaces

In this section we consider an attempt to prove conjecture 2 by using a degeneration of $K3$ surfaces to a union of planes and the blow-up of a $K3$ along points. Let Δ be an open disk and let X be the blow-up of $S \times \Delta$ along b general points of $S \times \{0\}$. The threefold X is equipped with two projections p_1, p_2 on Δ and S respectively and the general fiber X_t of p_1 is isomorphic to S, while X_0 is a reducible surface given by the union of b planes with a surface $\mathbb{S}$. The last surface is the blow-up of S along the b points. Each one of the b planes $\mathbb{P}_i$ cuts a curve R_i on $\mathbb{S}$ which is a line in $\mathbb{P}_i$ and a (-1)-curve in $\mathbb{S}$. Now given a line bundle L on S it is possible to construct infinitely many line bundles (depending on the integer k) $\mathcal{O}_X(L,k) := p_2^*(L) \otimes \mathcal{O}_X(k\mathbb{S})$ on X such that each one restricted to X_t gives L. Defining $\mathcal{X}(L,k)$ as the restriction to X_0 we have that

$$\begin{aligned} \mathcal{X}(L,k)_{|\mathbb{P}^i} &= \mathcal{O}_{\mathbb{P}^2}(k) \\ \mathcal{X}(L,k)_{|\mathbb{S}} &= \mathfrak{b}^*(L) \otimes \mathcal{O}_{\mathbb{S}}(-\textstyle\sum_{i=1}^b kE_i), \end{aligned}$$

where $\mathfrak{b} : \mathbb{S} \to S$ is the blow-up map. This construction allows us to degenerate a system on S to a union of systems on the $\mathbb{P}_i$'s and S in the following way. Let $Z := m_1q_1 + \cdots + m_rq_r$ be a subscheme of S with points in general position. Chosen $a_1, \ldots, a_b$ positive integers such that $a_1 + \cdots + a_b \leq r$, let Z_i' be the specialization of a_i points of Z to points of $\mathbb{P}_i$ (with the same multiplicities). Let $Z_{\mathbb{S}}'$ be the residual subscheme, made of $r - \sum a_i$ general points of $\mathbb{S}$. Given $Z' := Z_1' + \ldots + Z_b' + Z_{\mathbb{S}}'$, one has that $\mathcal{X}(\mathcal{L},k) \otimes \mathcal{I}_{Z'}$, is a degeneration of $\mathcal{L} \otimes \mathcal{I}_Z$. In this way, the starting system $\mathcal{L}$ through r degenerate to the system $\mathcal{L}_0$ on X_0 made by the $\mathcal{L}^i$ on the $\mathbb{P}_i$ and by the $\mathcal{L}_{\mathbb{S}}$ on $\mathbb{S}$. Observe that the last system corresponds to a system on S through less than r points. In this way, by using the fact that the homogeneous planar systems $\mathcal{L}_2(d, m^4)$, $\mathcal{L}_2(d, m^9)$ are never special, it is possible to use the preceding degeneration in an inductive way. So, for example consider the system $\mathcal{L}^n(d, m^{4^h})$, take $b = 4^{h-1}$ and put four general points on each of the $\mathbb{P}_i$. In this way the speciality of the starting system is related to that of $\mathcal{L}^n(d, m^{4^{h-1}})$ and so on. More generally we have the following (see [5]).

THEOREM 1. *If $\mathcal{L}^n(d,m)$ is non-special for all non-negative integers (d, m) then $\mathcal{L}^n(d', m'^{4^h 9^k})$ is non-special for all non-negative integers (d', m', h, k).*

Unfortunately it is an open problem to evaluate if a system through just one point is special or not. The only known example is $\mathcal{L}^4(d, 2d)$ as stated in Conjecture 2.

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ON SMOOTH SURFACES IN $\mathbb{P}^4$ CONTAINING A PLANE CURVE AND AN APPLICATION

Abstract. We consider smooth surfaces in $\mathbb{P}^4$ and we prove that, under cerain hyphotheses, this surfaces actually contain a plane curve. Then we prove that the degree of such surfaces is bounded. This yields a result on codimension two smooth subcanonical subvarieties in $\mathbb{P}^n$, $n \geq 5$ giving further evidence to Hartshorne conjecture in codimension two.

This is a short summary of the contents of two papers, see [1] and [2].
We work over an algebraically closed field of characteristic zero.
The main results are:

THEOREM 1. *Let $\Sigma \subset \mathbb{P}^4$ be an hypersurface of degree s with a (s-2)-uple plane, then the degree of smooth surfaces $S \subset \Sigma$ with $q(S) = 0$ is bounded.*

THEOREM 2. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $q(S) = 0$ and lying on a quartic hypersurface Σ, such that $Sing(\Sigma)$ has dimension two, then $d = deg(S) \leq 40$.*

As an application to codim. two subvarieties in $\mathbf{P}^n$ we have:

THEOREM 3. *Let $X \subset \mathbf{P}^n$, $n \geq 5$, be a smooth codimension two subcanonicalsubvariety, lying on a hypersurface Σ of degree s having alinear subspace K of codimension two and multiplicity $(s-2)$.Then X is a complete intersection.*

The proofs of the above results can be found in [1] and rest on a careful inspection of the geometric set up.

The assumptions of theorems 1 and 2 may be explained by next lemma.

LEMMA 1. *If $S \subset \Sigma \subset \mathbb{P}^4$ is a smooth surface, Σ a degree s hypersurface with a (s-2)-uple plane, then S contains a plane curve or $h^0(\mathcal{I}_S(2)) \neq 0$.*

From now on we suppose $h^0(\mathcal{I}_S(2)) = 0$ and thus S contains a plane curve.
As for theorem 3, recall that, by Lefschetz's theorem, if $X \subset \mathbf{P}^n$, $n \geq 4$, is a codimension two subvariety contained in a hypersurface Σ, if X is not a complete intersection, then $dim(X \cap Sing(\Sigma)) \geq n - 4$. We then consider a very particular situation: we assume the singular locus of Σ is as large as possible (codim. two) but the simplest possible (a linear subspace).

REMARK 1. (i) Theorem 2 is of some interest for the classification of non general type surfaces in $\mathbb{P}^4$, since it is known that such surfaces lie on hypersurfaces of low degree.

The idea of studying surfaces containing a plane curve is due to the fact that all known rational surfaces contain a plane curve (this has been observed by Catanese and Hulek). One could wonder if this can be generalized to non general type surfaces. The answer in negative, indeed there are sections of the Horrocks-Mumford bundle that do not contain any plane curve.
(ii) Theorem 3 gives further evidence to Hartshorne conjecture in codim. two.
(iii) It is easy to show that the assumption $q(S) = 0$ implies that all hyperplane sections of S are linearly normal in $\mathbf{P}^3$. It follows that all hyperplane sections of Σ have to be linearly normal too.
In the case of quartic hypersurfaces with $dim(Sing(\Sigma)) = 2$, this implies that $Sing(\Sigma)$ is a plane or a union of planes. This explains the difference between the hypotheses of theorems 1 and 2. Moreover this also explains why we did not start considering hypersurfaces with a linear subspace of codim. two and multiplicity $(s-1)$. Indeed the $\mathbf{P}^3$ section of such hypersurfaces is not linearly normal.

Let us fix some notations. Let S be a smooth surface in $\mathbb{P}^4$, let P be a plane curve contained in S, $p = deg(P)$, and Π be the plane containing P. We suppose P is the 1-dim. part of $S \cap \Pi$.
We denote by δ the linear system cut out on S, residually to P, by the hyperplanes containing Π, it turns out that $\delta = |H - P|$. Let $\mathcal{B}$ be the base locus of δ. We call $Y_H \in \delta$ the element cut by H and $C_H = S \cap H = Y_H \cup P$.

LEMMA 2. *The curve P is reduced and the base locus $\mathcal{B}$ of δ is empty or 0-dimensional and contained in Π. The general $Y_H \in \delta$ is smooth out of Π and doesn't have any component in Π.*

Proof. Clearly $\mathcal{B} \subset \Pi$. Let P_1 be an irreducible component of P, $P_1 \subset \mathcal{B}$. Then for all H containing Π, $C_H = H \cap S$ is singular along P_1. It follows that $T_x S \subset H$, $\forall x \in P_1$ and $\forall H \supset \Pi$ (S is smooth). We get $T_x S = \Pi$, $\forall x \in P_1$, but this contradicts Zak's theorem (see [4]) which states that the Gauss map is finite. The same argument shows that P is reduced. We conclude by Bertini's theorem. □

REMARK 2. Since δ is a pencil and $dim(\mathcal{B}) \leq 0$, it follows that $deg(\mathcal{B}) = (H - P)^2 = d - 2p + P^2$.
It turns out also that $\mathcal{B}$ is the residual scheme of $S \cap \Pi$ with respect to P.

For the proof of 3 we also need the following results.

LEMMA 3. *Let $X \subset \boldsymbol{P}^5$ be a smooth subcanonical 3-fold of degree d, then if $d \leq 25$, X is a complete intersection.*

LEMMA 4. *With usual notations, if $S \subset \mathbb{P}^4$ is subcanonical with $\omega_S \cong \mathcal{O}_S(a)$, $P \subset S$ a plane curve:*
(i) $deg(P) \leq a + 3$;
(ii) If $\mathcal{R} = \emptyset$, then S is a complete intersection.

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MULTIPOINT SESHADRI CONSTANTS ON $\mathbb{P}^2$

Abstract. Working over **C** and formalizing and sharpening approaches introduced in [12], [9] and [8], we give a method for verifying when a divisor on a blow up of $\mathbb{P}^2$ at general points is nef. The method is useful both theoretically and when doing computer computations. The main application is to obtaining lower bounds on multipoint Seshadri constants on $\mathbb{P}^2$. In combination with methods developed in [4], significantly improved explicit lower bounds are obtained.

Given a positive integer n, the Seshadri constant for points $p_1, \ldots, p_n$ of $\mathbb{P}^2$ is the real number

$$\varepsilon(N, p_1, \ldots, p_n) = \inf\left\{\frac{\deg(C)}{\Sigma_{i=1}^n \mathrm{mult}_{p_i} C}\right\},$$

where the infimum is taken with respect to all curves C, through at least one of the points. We also take $\varepsilon(n)$ to be defined as $\sup\{\varepsilon(p_1, \ldots, p_n)\}$, where the supremum is taken with respect to all choices of n distinct points p_i of $\mathbb{P}^2$ (see [6], [2] and [11]). It is well known and not difficult to prove that $\varepsilon(n) \leq 1/\sqrt{n}$, with equality if n is a square. Also, by results of Nagata [6], $\varepsilon(n)$ is known for $n < 10$, and, when $n \geq 10$ is not a square, Nagata [7] conjectured that $\varepsilon(n) = 1/\sqrt{n}$. Although this conjecture has not yet been verified for any $n \geq 10$ not a square, the general belief is that it is correct, hence the attention paid here and elsewhere to obtaining lower bounds for $\varepsilon(n)$, focusing in the case $n \geq 10$.

Here, refining an approach of [9] and [10] (see also Tutaj-Gasińska's contribution to the present volume) which in turn refine and extend the method used in [12], we give a method that provides a basis for obtaining arbitrarily accurate estimates of $\varepsilon(n)$, which we apply to obtain lower bounds for $\varepsilon(n)$ which for almost all n improve on the bounds cited above. Let us denote by $\alpha(m, p_1, \ldots, p_n)$ (respectively, $\alpha_0(m, p_1, \ldots, p_n)$) the least degree of a curve (respectively, irreducible curve) passing with multiplicity at least m (respectively, exactly m) through each point p_i. If the points are in general position in $\mathbb{P}^2$, we write simply $\alpha(m^{[n]})$ and $\alpha_0(m^{[n]})$. Our method involves two steps. The first step shows how to convert estimates of values of α to bounds on $\varepsilon(n)$. The second step, based on our work in [4], concerns actually making the estimates of the values of α.

To provide a basis for making comparisons of different lower bounds on $\varepsilon(n)$, it is convenient to write them in the form $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/f(n)})$, where f is a function of n. Note that the larger $f(n)$ is, the better is the bound.

THEOREM 1. *Let $n \geq 10$ be an integer, and $\mu \geq 1$ a real number.*

1. *If $\alpha(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ for every integer $1 \leq m < \mu$, then $\varepsilon(n) > \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{(n-2)\mu}}$.*

2. *If* $\alpha_0(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ *for every integer* $1 \leq m < \mu$, *and if* $\mu \leq 6(n-1)$, *then* $\varepsilon(n) \geq \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{n\mu}}$.

The basic tool for the proof of Theorem 1 (to be found in [5]) is the study of *abnormal* curves, i.e., irreducible counterexamples to Nagata's conjecture. From the properties of the intersection product on general blowups of $\mathbb{P}^2$ we obtain restrictions on abnormal curves; using these, for every $t < 1/\sqrt{n}$ we can produce an explicit finite list of tuples $(d, m_1, \ldots, m_n)$ such that if $\varepsilon(n) < t$ then for some degree d and multiplicities m_i on the list there exists an abnormal curve $C(d, m_1, \ldots, m_n)$ whose degree and multiplicities are one of the entries of the list, and thus $\varepsilon(n) = d/(m_1 + \cdots + m_n)$. So to conclude that $\varepsilon(n) \geq t$ it is enough to show that each tuple on the list does *not* correspond to an irreducible plane curve. For any specific n, our best lower bound on $\varepsilon(n)$ is obtained by direct application of this method. For each nonsquare $10 \leq n \leq 58$, Table 1 gives the best value we know for $f(n)$ (truncated to two decimals), along with a possible abnormal curve $C(d, m^{[n]})$ which we are unable to rule out but which would have to be ruled out in order to verify a larger value for $f(n)$.

This direct approach is algorithmic; by analyzing the algorithm, based on our work in [4], we are also able to give weaker but explicit lower bounds in terms of n.

COROLLARY 1. *Let* $n > 16$ *be a nonsquare integer, let* $d = \lfloor\sqrt{n}\rfloor$ *and consider* $\Delta = n - d^2 > 0$. *Let us define*

$$f(n) = \begin{cases} n(n-1) & \text{if } \Delta = 2, \\ n(n - 3\sqrt{n} - 4)/2 & \text{if } \Delta > 2 \text{ is even}, \\ n(n - 3\sqrt{n} - 2) & \text{if } \Delta \text{ is odd and } \Delta < 4\sqrt[4]{n} + 1, \\ n^2 & \text{if } \Delta \text{ is odd and } 2d - 1 > \Delta \geq 4\sqrt[4]{n} + 1, \\ n(n\sqrt{n} - 5n + 5\sqrt{n} - 1)/2 & \text{if } \Delta = 2d - 1; \end{cases}$$

then $\varepsilon(n) \geq \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{f(n)}}$.

Perhaps the best previous general bound for $n \geq 10$ is given in [10], for which $f(n) = 12n + 1$. As Corollary 1 shows, for our bounds $f(n)$ is at least quadratic in n, so for n large enough (indeed, for $n \geq 59$), our bounds involve larger values of $f(n)$. For special values of n, [1] also gives bounds better than those of [10], and these bounds are also quadratic in n. However, except when $n - 1$ is a square, our bounds are better, for n large enough. Bounds are also given in [3], which are almost always better than any bound for which $f(n)$ is linear in n. Although these bounds are not simple enough to make comparisons easy, computations for specific values of n show in almost all cases that the bounds we obtain here are better than those of [3]. The results shown in Table 1 for $n - 1$ a square and for $n = 19, 22$ are given by [1] and are as good or better than what we obtain; the result for $n = 41$ comes from [3]; all other results shown in Table 1 are better than what was known previously.

n	f	C(d,m[n])	n	f	C(d,m[n])	n	f	C(d,m[n])
10	886.62	C(256,81)	27	997.96	C(161,31)	43	1741.5	C(236,36)
11	402.28	C(106,32)	28	1304.25	C(201,38)	44	1985.5	C(252,38)
12	300.52	C(83,24)	29	639.45	C(113,21)	45	3782.25	C(275,41)
13	325	C(90,25)	30	1230.76	C(219,40)	46	3140.26	C(217,32)
14	740.6	C(86,23)	31	1093.26	C(128,23)	47	7109.17	C(994,145)
15	566.78	C(89,23)	32	940.52	C(147,26)	48	1521.39	C(187,27)
17	1089	C(136,33)	33	1093.55	C(178,31)	50	9801	C(700,99)
18	466.94	C(89,21)	34	1731.93	C(239,41)	51	3313.98	C(407,57)
19	28900	C(5928,1360)	35	974.47	C(136,23)	52	6257.33	C(274,38)
20	660.64	C(143,32)	37	5329	C(444,73)	53	3499.89	C(313,43)
21	1187.1	C(142,31)	38	1898.97	C(265,43)	54	5713.2	C(338,46)
22	38809	C(7392,1576)	39	1779.7	C(231,37)	55	2370.64	C(304,41)
23	576	C(115,24)	40	1601.66	C(196,31)	56	3193.01	C(419,56)
24	1009.2	C(142,29)	41	1025	C(160,25)	57	2608.42	C(234,31)
26	2601	C(260,51)	42	1306.94	C(149,23)	58	9802	C(396,52)

Table 1: Current best known values of $f(n)$ for nonsquares $10 \le n \le 58$

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EQUISINGULAR FAMILIES OF CURVES

Abstract. In this note we refer to results on conditions which ensure that equisingular families of curves on smooth projective surfaces are non–empty, respectively T-smooth, respectively irreducible.

In the early 20th century Severi asked in his book [14] the question when the family $V_d^{irr}(r \cdot A_1)$ of irreducible plane curves of degree d having r simple nodes as only singularities is smooth of the expected dimension. We note that a plane curve of degree d is given by a homogeneous polynomial of degree d in three variables, which is determined up to scalar multiple, and since a node imposes one condition we expect that the family has dimension

$$\text{edim}\, V_d^{irr}(r \cdot A_1) = \frac{d \cdot (d+3)}{2} - r.$$

Severi proves that, whenever possible, the family has the desired property – more precisely, this is the case whenever

$$r \leq \frac{(d-1) \cdot (d-2)}{2},$$

where the difference of these two numbers is just the genus of the curve. In the same book Severi claims that the family is also irreducible, but his proof contains a gap and it took a long time before the proof finally was completed in [7].

The question of Severi has led to generalisations in several directions, considering nodal families of curves on other surfaces (cf. e. g. [16, 4, 3, 5]) or families with in the projective plane with more complicated singularities (cf. e. g. [1, 2, 6]). Already for cuspidal curves in the plane or for nodal curves on surfaces in $\mathbb{P}_{\mathbb{C}}^3$ we have no longer such a complete picture that the family is smooth of expected dimension and irreducible whenever it is non-empty as for nodal plane curves (cf. e. g. [17, 15, 4, 3]). The idea, therefore, can only be to find numerical criteria which ensure that the family is

1. non-empty,
2. smooth of the expected dimension, respectively
3. irreducible.

In our research we have investigated families of curves with arbitrary singularities on several classes of smooth projective surfaces, including generic surfaces in $\mathbb{P}_{\mathbb{C}}^3$,

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generic K3-surfaces, geometrically ruled surfaces and generic products of curves. Allowing arbitrary singularities we can hardly expect to get conditions which are sharp as e. g. in [4] in the case of nodal curves on surfaces in $\mathbb{P}_{\mathbb{C}}^3$. Instead it is our aim to find conditions which are "asymptotically" of the right type. In order to explain this we need to make the setting a bit more precise. For an elaborate definition of the notation used we refer to [8].

Let Σ be a smooth projective surface, L some ample divisor on Σ, and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ topological respectively analytical singularity types. We denote by

$$V_d^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$$

the subfamily of the linear system $|dL|$ of irreducible curves with precisely r singular points of type $\mathcal{S}_1, \ldots, \mathcal{S}_r$, e. g. if $\mathcal{S}_1 = \ldots = \mathcal{S}_r = A_1$ is the class of an ordinary node we are considering the subfamily of r-nodal curves in $|dL|$. It is our aim to find sufficient numerical conditions of the form

$$\sum_{i=1}^{r} f\big(\operatorname{inv}(\mathcal{S}_i)\big) \leq g(d)$$

where inv is some invariant of the singularity type and f and g polynomial functions. That the condition is "asymptotically" of the right type means that we cannot possibly achieve a condition involving the same invariant but where the degree of f is larger respectively the degree of g smaller, e. g. since there is a necessary condition involving the same invariant and where the polynomial functions have the same degrees. As already indicated by the case of nodal plane curves the irreducibility seems to be the hardest question.

In order to keep the exposition short I will just outline the results for K3-surfaces with Picard number one and for analytical singularity types.

In [12] we give an answer to the question of the existence of curves with prescribed singularities in $|dL|$ combining an h^1-vanishing result with the Viro gluing method and get the condition

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq \frac{1}{18} \cdot (d-1)^2 \cdot L^2,$$

where μ denotes the Milnor number of $\mathcal{S}_i$. Due to the genus formula this condition is asymptotically of the right type.

In [11] we show that $V_d^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is smooth of the expected dimension if

$$\sum_{i=1}^{r} \gamma_1^{ea}(\mathcal{S}_i) < d^2 \cdot L^2,$$

where γ_1^{ea} is a new invariant introduced in [13]. The proof comes down to an h^1-vanishing result which is proved using the technique of Bogomolov instability. Moreover, we show in [10] that for ordinary multiple points the condition is asymptotically of the right type.

In [9] we produce a sufficient condition for the irreducibility of $V_d^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$, again using Bogomolov instability in order to provide a “generic” h^1-vanishing. The condition looks like

$$\sum_{i=1}^{r} \big(\tau(\mathcal{S}_i) + 2\big)^2 < \frac{54(L^2)^2 + 72L^2}{(11L^2 + 12)^2} \cdot d^2 \cdot L^2.$$

This is, as far as we know, the best known result in this direction. But so far it is not clear whether it has the right asymptotics.

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A CONJECTURE ON SPECIAL LINEAR SYSTEMS OF $\mathbb{P}^3$

Abstract. In this note we deal with linear systems of $\mathbb{P}^3$ through fat points. We consider the behavior of these systems under a cubo-cubic Cremona transformation that allows us to produce a class of special systems which we conjecture to be the only ones.

1. Introduction and statement of the problem

In what follows we assume that the ground field is algebraically closed of characteristic 0. Consider r points in general position on $\mathbb{P}^n$, to each one of them associate a natural number m_i called the *multiplicity* of the point. We will denote by $\mathcal{L} = \mathcal{L}_n(d, m_1, \ldots, m_r)$ the linear system of hypersurfaces of degree d through the r points with the given multiplicities. Define the *virtual dimension* of the system as $v(\mathcal{L}) = \binom{d+n}{n} - \sum \binom{m_i+n-1}{n} - 1$ and its *expected dimension* by $e = \max\{v, -1\}$. Observe that $e \leq \dim(\mathcal{L})$ and that the inequality may be strict if the conditions imposed by the points are dependent. In this case we say that the system is *special*. The problem of classifying special systems has been completely solved in the case $m_1 = \ldots = m_r = 2$ (see [1]) and it has been largely studied for linear systems on the plane (see [3, 4, 10]). The main conjecture on the structure of special planar systems has been formulated in [6, 7]. In this note we report about some recent results in the case of $\mathbb{P}^3$. In [8] we gave a counterexample to a conjecture (see [2]) about the structure of special linear systems of $\mathbb{P}^3$. Starting from this idea in [9] we analyzed the behavior of linear systems under a cubo-cubic Cremona transformation of $\mathbb{P}^3$. This allowed us to construct a class of special linear systems which we conjecture to be all the possible ones.

Throughout this note we will denote by X the blow up of $\mathbb{P}^3$ along the r fixed points and by $E_i \cong \mathbb{P}^2$ the exceptional divisors.

From the Riemann-Roch formula on a smooth threefold, we obtain that $v(\mathcal{L}) = (\mathcal{L}(\mathcal{L} - K_X)(2\mathcal{L} - K_X) + c_2(X)\mathcal{L})/12$. If the linear system can be written as $\mathcal{L} = F + \mathcal{M}$, where F is a fixed divisor of $\mathcal{L}$ and $\mathcal{M}$ is the residual system, then the above formula implies:

$$v(\mathcal{L}) = v(\mathcal{M}) + v(F) + \frac{F\mathcal{M}(\mathcal{L} - K_X)}{2}. \tag{1}$$

All the results described in this note can be found in [9].

2. Cubic Cremona transformations and linear systems

It is possible to consider the behavior of linear systems under a birational transformation of $\mathbb{P}^3$. In particular we need to consider a transformation which sends linear systems through points into systems through points. Consider the system $\mathcal{L}_3(3, 2^4)$; by putting the four double points in the fundamental ones, the associated rational map is $\mathrm{Cr} : (x_0 : x_1 : x_2 : x_3) \to (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1})$. This birational map induces the following action on a linear system $\mathcal{L}$:

$$\mathrm{Cr}\,(\mathcal{L}) = \mathcal{L}_3(d+k, m_1+k, \dots, m_4+k, m_5, \dots, m_r), \tag{2}$$

where $k = 2d - \sum_{i=1}^{4} m_i$. By using this transformation, it is easy to see that if $2d < m_1 + m_2 + m_3$, then the plane through the first three points is a fixed component of the system. Observe that $\dim \mathrm{Cr}\,(\mathcal{L}) = \dim \mathcal{L}$ but in general the virtual dimensions of the two systems may be different as stated in the following:

PROPOSITION 1. *Let $\mathcal{L}$ be a linear system such that $2d \geq m_i + m_j + m_k$ for any choice of $\{i, j, k\} \subset \{1, 2, 3, 4\}$ then*

$$v(\mathrm{Cr}\,(\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \binom{1 + t_{ij}}{3} - \sum_{t_{ij} \leq -2} \binom{1 - t_{ij}}{3}, \tag{3}$$

where $t_{ij} = m_i + m_j - d$.

In particular this implies that $v(\mathrm{Cr}\,(\mathcal{L})) \geq v(\mathcal{L})$ if the degree of $\mathrm{Cr}\,(\mathcal{L})$ is smaller than the one of $\mathcal{L}$. This means that as far as $2d < m_1 + m_2 + m_3 + m_4$ we can perform a Cremona transformation decreasing the degree and the multiplicities of the system. If at some step we get a system such that $2d < m_1 + m_2 + m_3$, then we remove the plane from the base locus. After a finite number of steps we get a system with $2d \geq m_1 + m_2 + m_3 + m_4$, which we say to be in *standard form*.

3. Conjecture

To each linear system $\mathcal{L}$ we associate the 1-cycle $\Gamma(\mathcal{L}) := \sum_{t_{ij} \geq 1} t_{ij} l_{ij}$, where $t_{ij} = m_i + m_j - d$ and l_{ij} is the line through p_i and p_j. Observe that by definition $H^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = H^0(\mathcal{L})$, since each line $l_{ij} \in \Gamma(\mathcal{L})$ is contained in $\mathrm{Bs}\,(\mathcal{L})$ with multiplicity at least t_{ij}.

PROPOSITION 2. *The relation between the Euler characteristic of the two sheaves is given by:*

$$\chi(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = \chi(\mathcal{L}) + \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3}.$$

This implies that for each $\Gamma = \sum \alpha_{ij} l_{ij}$ with $2 \le \alpha_{ij} \le t_{ij}$ we have

$$\dim \mathcal{L} - v(\mathcal{L}) \ge \sum \binom{\alpha_{ij}+1}{3} - h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma).$$

It is possible to prove that if Γ is just a multiple line then $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) = 0$ and the system is special.

The preceding discussion gives us a class of special systems in standard form. We can construct another such class in the following way. Let $Q = \mathcal{L}_3(2, 1^9)$ be a quadric through nine points and suppose that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$. By formula 1 we have that $v(\mathcal{L}) < v(\mathcal{L} - Q)$ which implies that $\mathcal{L}$ is special. By assuming the Harbourne-Hirschowitz conjecture to be true for planar systems with 10 points, it is possible to prove that if $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$ then Q is a fixed component of $\mathcal{L}$.

CONJECTURE 2. A linear system $\mathcal{L}$ in standard form is special if and only if one of the following holds:

(i) there exists a quadric Q such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$;

(ii) at least one of the coefficients of $\Gamma(\mathcal{L})$ is bigger then 1.

Assuming that this conjecture and the Harbourne-Hirschowitz (for systems through 10 points) are true we can remove all the quadrics of step (i) from the base locus of $\mathcal{L}$. Then the residual system $\mathcal{L}'$ is still in standard form and

$$\dim \mathcal{L} = v(\mathcal{L}') + \sum_{t'_{ij} \ge 2} \binom{t'_{ij}+1}{3}$$

assuming that $h^2(\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')}) = 0$.

We conclude this note with two propositions about *homogeneous* linear systems, i.e. the systems $\mathcal{L}$ for which $m_1 = \ldots = m_r = m$.

PROPOSITION 3. *The system* $\mathcal{L}$ *is empty for* $d \le 2m - 1$ *and* $r \ge 8$.

By assuming Conjecture 2 and Harbourne-Hirschowiz conjecture for linear systems on $\mathbb{P}^2$ with 10 points, we can also prove the following:

PROPOSITION 4. *If* $d \ge 2m$ *the system* $\mathcal{L}$ *is special if and only if* $r = 9$ *and* $2m \le d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$.

Therefore if the system $\mathcal{L}$ has more than 9 fixed points (or exactly 8 points) then it is not special. If it has 9 fixed points, it is special if and only if d satisfies the inequalities of the preceding proposition. If $r \le 7$ and $d \ge 2m$, the system can not be special. Finally, if $r \le 7$ and $d \le 2m - 1$, by applying a finite number of Cremona transformations we reduce to a system in standard form.

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A BOUND FOR SESHADRI CONSTANT ON $\mathbb{P}^2$

Abstract. This is a summary of the talk given in Workshop on Polynomial Approximation and Projective Embeddings, Torino, September 17-18, 2003. The talk was based on the article, published electronically on 28 July 2003 in Math. Nachr. **257**.

1. Theorems

Let C be a curve in $\mathbb{P}^2$ passing through general points $P_1, \ldots, P_r$ with multiplicities $m_1, \ldots, m_r$. The conjecture stated by Nagata in [4] says that for $r > 9$ it holds: $d := \deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i$. The conjecture is still open except for the case when r is a square, cf. [4], [3].

The above question may be restated using Seshadri constants (cf. [2]). The multiple point Seshadri constant of the line bundle $\mathcal{O}_{\mathbf{P}^2}(1)$, is defined as follows. For $P_1, \ldots, P_r$, pairwise distinct points on $\mathbb{P}^2$ we define $\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1); P_1, \ldots, P_r) := \inf_C \left\{ \frac{\deg C}{\sum_{i=1}^{r} m_i} \right\}$, where C is a curve through $P_1, \ldots, P_r$ with $\mathrm{mult}_{P_i} C = m_i$, $i = 1 \ldots r$. So, Nagata's conjecture says that $\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1), P_1, \ldots, P_r) = \frac{1}{\sqrt{r}}$ for general points $P_1, \ldots, P_r$. We prove:

THEOREM 1. *For $P_1, \ldots, P_r$, $r > 9$, general points on $\mathbb{P}^2$, we have* $\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1), P_1, \ldots, P_r) \geq \frac{1}{\sqrt{r + \frac{1}{12}}}$.

The theorem given below gives an ampleness criterion, crucial in the proof of Theorem 1.

THEOREM 2. *Let $\pi : X \to \mathbf{P}^2$ be a blowing up of $\mathbb{P}^2$ in $r > 9$ general points, $P_1, \ldots, P_r$, with exceptional divisors $E_1, \ldots, E_r$. Let $H := \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$. Consider a line bundle on X of the form $L := dH - k \sum_{i=1}^{r} E_i$, where d and k are natural numbers with $d \geq 3k + 1$. Assume that $r \leq \frac{d^2}{k^2} - \frac{1}{12}$. Then L is ample on X.*

2. About proofs

Let us say a few words about the proof of Theorem 2. This proof is based upon three results. First of them is the result of Ciliberto and Miranda, [1]:

RESULT 1. Consider a linear system on $\mathbb{P}^2$ of curves of degree p passing through $r > 9$ general points with multiplicity exactly m. Then, for $m \leq 12$ such a system has the expected dimension.

The second result is by Xu, [6]:

RESULT 2. Let $P_1, \dots, P_{r-1}, P$ be general points in $\mathbb{P}^2$ and let C be a reduced and irreducible curve of degree p passing through P_i's with multiplicities $\mathrm{mult}_{P_i} C = m_i$, for $i = 1, \dots, r-1$ and through P with multiplicity $m \geq 2$. Then $p^2 - \sum_{i=1}^{r-1} m_i^2 - m_j \geq m^2 - m + 1$.

The next result was proved by Szemberg, [5].

RESULT 3. Let $P_1, ..., P_r$ be general points on $\mathbb{P}^2$, let a curve C be reduced, irreducible and submaximal (i.e. such that $\frac{\deg C}{\sum_{i=1}^r \mathrm{mult}_{P_i} C} < \sqrt{\frac{1}{r}}$.) Then C is almost homogeneous (i.e. all but at most one multiplicities at P_i's equal).

Having these three results, the idea of the proof is simple. As $L^2 > 0$ we have to prove that for every reduced and irreducible curve C it holds $LC > 0$. To obtain this, assume that $C = pH - \sum_{i=1}^r m_i E_i$, with $m_1 \geq ... \geq m_r \geq 0$. The first step of our proof is to check, using Result 1, that for C with $m_r > 12$, $LC > 0$ holds. Next, if for C we have $p\sqrt{r} \geq \sum_{i=1}^r m_i$, then of course $LC > 0$. Thus, assume that $p\sqrt{r} < \sum_{i=1}^r m_i$ (so C is a submaximal curve for $\mathcal{O}_{\mathbf{P}^2}(1)$). We have to check that for every such curve C, $LC > 0$; from Result 3 it follows that it is enough to consider C almost homogeneous. From Result 1 it follows that we can exclude C homogeneous. This way we are left with: $C = pH - \sum_{i=1}^{r-1} m_1 E_i - m_r E_r$ or $C = pH - \sum_{i=2}^r m_r E_i - m_1 E_1$. Analyzing carefully the cases we prove the theorem.

Now we present the proof of Theorem 1: let C be a curve of degree p passing through $P_1, ..., P_r$ with multiplicities $m_1, ..., m_r$ (and at least one m_i is greater than zero). From the definition of Seshadri constants it follows that we have to prove that $\frac{p}{\sum_{i=1}^r m_i} \geq \frac{1}{\sqrt{r+\frac{1}{12}}}$. To show this, take L as in Theorem 2, with given r and k and with minimal possible $d_k = \left\lceil k\sqrt{r + \frac{1}{12}} \right\rceil$. Then, $LC = d_k p - k \sum_{i=1}^r m_i > 0$, as L is ample. So, $p > \frac{k}{d_k} \sum_{i=1}^r m_i$. Taking the limit with $k \to \infty$ we get $\frac{p}{\sum_{i=1}^r m_i} \geq \frac{1}{\sqrt{r+\frac{1}{12}}}$.

REMARK 1. The presented bound is already meaningly improved by the recent results of Harbourne and Roé, cf. arXiv.org: math/0309064.

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LINEAR SYSTEMS OF PLANE CURVES WITH BASE POINTS OF BOUNDED MULTIPLICITY

1. Introduction

Let $\mathcal{L} = \mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ denote the linear system of degree d plane curves with k_i base points of multiplicity m_i for $i = 1, \ldots, s$, all in general position. The virtual and expected dimensions of $\mathcal{L}$ are respectively defined to be:

$$v(\mathcal{L}) := \binom{d+2}{2} - \sum k_i \binom{m_i+1}{2} - 1 \tag{1}$$

$$e(\mathcal{L}) := \max\{v(\mathcal{L}), -1\}. \tag{2}$$

The Harbourne-Hirschowitz conjecture gives geometric meaning to when multiple base points in general position fail to impose independent linear conditions on the space of degree d plane curves; i.e., when the dimension of $\mathcal{L}$ is greater than expected. The main result of [4] is a verification of this conjecture if the multiplicities of the base points are bounded by 7.

THEOREM 1. *If $m_i \leq 7$ for $i = 1, \ldots, s$, then $\dim \mathcal{L} > e(\mathcal{L})$ if and only if its base locus of $\mathcal{L}$ contains a multiple copy of a (-1)-curve.*

Theorem 1 follows from the lemma below, which reduces the proof the theorem to a finite, but very large, number of cases. Most of these cases are handled using a computer, the rest with ad hoc methods.

LEMMA 1. *For any positive integer M, there exists $D = D(M)$ with the following property: if the Harbourne-Hirschowitz conjecture is true for all $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ with $m_i \leq M$ for $i = 1, \ldots, s$ and $d < D(M)$, then it is true for all $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ with $m_i \leq M$ for $i = 1, \ldots, s$ and all values of d.*

The table below shows the first few values of $D(M)$.

M	2	3	4	5	6	7	8	9	10	...	N
$D(M)$	9	13	17	21	25	29	34	42	51	...	$O(N^2)$

Table 12.1: Values of $D(M)$ for $M = 2, \ldots, 10$

The proof Lemma 1 is similar to the proof of Theorem 4.1 in [3], which uses a degeneration of $\mathbb{P}^2$ into a reducible surface consisting of two rational components. This yields a recursive formula for the dimension of $\mathcal{L}$. For details, see [4].

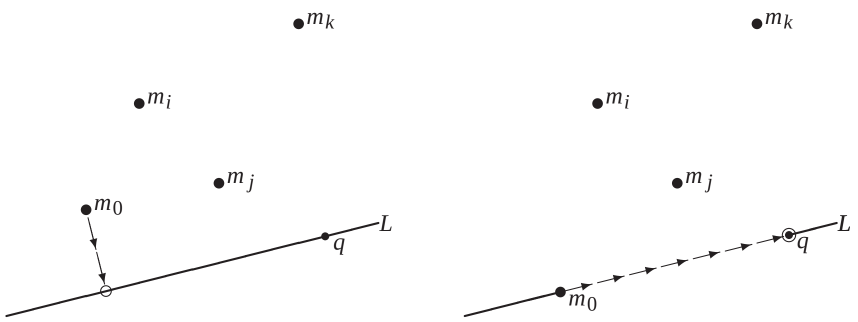

Figure 1: The two-step process of specializing the fat point m_0

2. Aligned ideals

The main algorithm used by the computer program arises from specializing the base points of $\mathcal{L}$ onto a fixed line $L \subseteq \mathbb{P}^2$, and then along the line onto a fixed point $q \in L$. After we specialize all of the base points of $\mathcal{L}$ in this manner, we are left with a linear system of plane curves with a rather exotic singularity at q, and our first goal is to describe the nature of these exotic singularities.

Choose coordinates in $\mathbb{P}^2$ such that

$$L = Z(Y), \tag{3}$$

$$q = Z(Y, Z). \tag{4}$$

In local coordinates $[x : y : 1]$, the line L is the x-axis and q is the point at infinity. Let α denote a strictly decreasing sequence of positive integers,

$$\alpha_1 > \alpha_2 > \cdots > \alpha_h > 0. \tag{5}$$

Let $\mathfrak{I}_d(\alpha)$ denote the d-th graded part of the ideal generated by the monomials $Y^i Z^{\alpha_i}$, for $i = 1, \ldots, h$. Any ideal of this form will be called an *aligned ideal*.

Aligned ideals are monomial ideals by definition, but not all monomial ideals are aligned—for example, $\langle X^2, Y^2 \rangle$ is not an aligned ideal. We can visualize α by creating a $(d + 1) \times (d + 1)$ triangle of boxes which represent the monomial basis for degree d polynomials, with Y^d representing the box in the top corner, Z^d and X^d respectively representing the bottom left and bottom right corner boxes, and the rest of the monomials distributed among the boxes in the usual manner. The polynomials in $\mathfrak{I}_d(\alpha)$ are generated by the monomials corresponding to all but the right-most α_i boxes in the i-th row from the bottom. If we shade in the boxes corresponding to monomials which do not lie in $\mathfrak{I}_d(\alpha)$, we see the "shape" of α appearing in the bottom right corner of boxes. For example, in Figure 2, the aligned ideal $\mathfrak{I}_3(3, 2)$ consists of cubics with a cusp point at q and is generated by the monomials Y^3, Y^2Z, YZ^2, XY^2, and Z^3.

These box diagrams are particularly convenient for denoting how an aligned ideal will change as we impose an additional m-fold point and then specialize the m-fold point

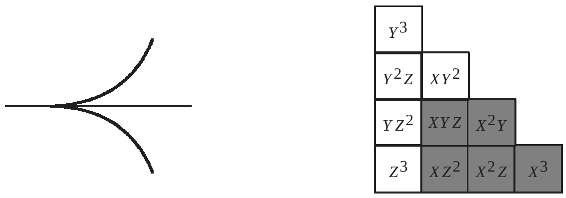

Figure 2: The aligned ideal $\mathfrak{I}_3(3, 2)$

onto L and q. The box diagram for the new aligned ideal has up to $\binom{m}{2}$ more shaded boxes than to the original one. The algorithm below exactly determines which boxes become shaded in this process.

1. Fill in the lowest $m + 1 - j$ unshaded boxes in the j-th column from the left, for $j = 1, \dots, m$, with dots. If there are not enough unshaded boxes in a column, we use as many dots as we need and discard the rest.
2. Slide each row of dots as far to the right as possible within the white boxes.
3. Shade in the dotted boxes.

The two horizontal rows of the Figure 3 below demonstrate this algorithm performed for $m = 3$ and two different aligned ideals. In the first row, $\mathfrak{I}_4(2, 1)$ becomes $\mathfrak{I}_4(5, 3, 1)$. In the second row, $\mathfrak{I}_4(3, 2, 1)$ becomes $\mathfrak{I}_4(5, 4, 2)$ while discarding a dot in the first step.

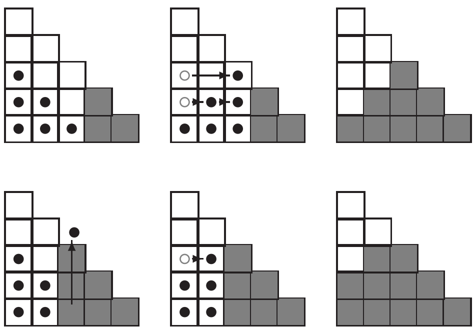

Figure 3: A couple of examples

By upper-semicontinuity, the dimension of $\mathcal{L}$ is bounded above by the dimension of the aligned ideal which arises from specializing all of the multiple points onto L and q; this is exactly one less than the number of white boxes left after we repeat the box diagram algorithm with all the multiplicities. Clearly, if it is possible to iterate the box diagram with all of the m_i (with multiplicity k_i) without losing any dots, then every multiple point imposes them maximum possible number of linear conditions, and so $\mathcal{L}$ is non-special.

3. The proof of the Theorem 1

To prove the Harbourne-Hirschowitz conjecture for $M \leq 7$, we programmed a computer to enumerate all linear systems $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ of degree 29 or less, with points of multiplicity 7 or less. There $125, 220, 076$ of these, almost all of which were shown to satisfy the Harbourne-Hirschowitz conjecture by the box diagram algorithm:

	125,220,076	total systems
-	124,850,912	are empty via the box diagram algorithm
-	366,691	are empty by Bezout's theorem (see [4])
-	2,013	contain multiple (-1)-curves in the base locus
-	418	are empty via a degeneration $\mathbb{P}^2$ (see [4])
	42	systems remain

The remaining 42 linear systems are found in Table 2 of [4]. A large number of these are either homogeneous or satisfy $d < m_1 + m_2 + m_3$ for m_1, m_2, and m_3 representing the three highest multiplicities of the points in the base locus; thus, the speciality of the linear system is equivalent to the speciality of one of lower degree via a quadratic Cremona transformation. The rest of the systems are handled case by case in [4] using elementary techniques.

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